

DYNAMICS ON NETWORKS I. COMBINATORIAL CATEGORIES OF MODULAR CONTINUOUS-TIME SYSTEMS

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ABSTRACT. We develop a new framework for the study of complex continuous time dynamical systems based on viewing them as collections of interacting control modules. This framework is inspired by and builds upon the groupoid formalism of Golubitsky, Stewart and their collaborators. Our approach uses the tools and—more importantly—the stance of category theory. This enables us to put the groupoid formalism in a coordinate-free setting and to extend it from ordinary differential equations to vector fields on manifolds. In particular, we construct combinatorial models for categories of modular continuous time dynamical systems. Each such model, as a category, is a fibration over an appropriate category of labeled directed graphs. This makes precise the relation between dynamical systems living on networks and the combinatorial structure of the underlying directed graphs, allowing us to exploit the relation in new and interesting ways.

CONTENTS

1. Introduction	1
2. Groupoid invariant vector fields on Euclidean spaces	5
3. Linear groupoid invariant vector fields	30
4. Groupoid invariant vector fields on manifolds	33
5. Groupoid invariance versus group invariance	41
6. Functoriality of limits	43
Appendix A. Elements of category theory	48
References	57

1. INTRODUCTION

1.1. Background. The goal of this paper is to develop a new framework for dynamics on networks using the techniques and methods of category theory. Networks, and the dynamical systems defined on them, are ubiquitous in science, engineering and the social sciences; understanding dynamics on networks constitutes a major scientific challenge with applications across a variety of fields. For example, the mathematical study of dynamics on networks plays an important role in the study and design of communications networks [1]; in cognitive science, computational neuroscience, and robotics (see, for example [2–7]); in the study of gene regulatory networks [8–10] and more general complex biochemical networks [11]; and finally in complex active media [12–16]. Current approaches to understanding dynamics on networks include ideas from statistical physics and random graph theory (see [17–24] and the many references therein).

We propose to develop a new approach to this problem which is inspired by and builds upon the groupoid formalism of Golubitsky, Stewart and their collaborators [25–51] and references therein). The key idea of the groupoid formalism is this: many networks are **modular**, and the modes of interaction between pieces of the network are **repeated** across the network. This repetition is

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a symmetry in a very broad sense of the word. In the case of network dynamics modeled by ordinary differential equations (ODEs), Golubitsky et al. found a mathematical formulation of this symmetry as a groupoid symmetry of the governing equations. We follow their lead and develop a new framework for the groupoid formalism using the tools and, perhaps more pertinently, the stance of category theory [52].¹

In this first paper in a series we put the groupoid formalism of Golubitsky et al. in a coordinate-free framework and then extend it from ordinary differential equations to vector fields on manifolds. In particular we construct combinatorial models for categories of modular continuous-time dynamical systems. Discrete time systems, groupoid-compatible numerical methods, and stochastic dynamical systems defined on networks will all be taken up in subsequent papers [53, 54].

To explain what our work is about we start with an example. Consider an ODE in $(\mathbb{R}^n)^3$ of the form

$$(1.1.1) \quad \dot{x}_1 = f(x_2), \quad \dot{x}_2 = f(x_1), \quad \dot{x}_3 = f(x_2)$$

for some smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. That is, consider the flow of the vector field

$$F : (\mathbb{R}^n)^3 \rightarrow (\mathbb{R}^n)^3, \quad F(x_1, x_2, x_3) = (f(x_2), f(x_1), f(x_2)).$$

It is easy to check that F is tangent to the diagonal

$$\mathbb{R}^n \simeq \Delta = \{(x_1, x_2, x_3) \in (\mathbb{R}^n)^3 \mid x_1 = x_2 = x_3\}$$

and that the restriction of the flow of F to Δ is the flow of the ODE

$$\dot{u} = f(u).$$

One can also see another invariant submanifold of F :

$$(\mathbb{R}^n)^2 \simeq \Delta' = \{(x_1, x_2, x_3) \in (\mathbb{R}^n)^3 \mid x_1 = x_3\}.$$

On Δ' the flow of F is the flow of the ODE

$$\dot{v}_1 = f(v_2), \quad \dot{v}_2 = f(v_1).$$

Moreover the projection

$$\pi : (\mathbb{R}^n)^3 \rightarrow \Delta', \quad \pi(x_1, x_2, x_3) = (x_1, x_2, x_1)$$

intertwines the flows of F on $(\mathbb{R}^n)^3$ and on Δ' . We have thus observed two subsystems of $((\mathbb{R}^n)^3, F)$ and three maps between the three dynamical systems:

$$(1.1.2) \quad (\Delta, F|_{\Delta}) \hookrightarrow ((\mathbb{R}^n)^3, F) \xrightleftharpoons[\pi]{} (\Delta', F|_{\Delta'})$$

Where do these subsystems and maps come from? There is no obvious symmetry of $(\mathbb{R}^n)^3$ that preserves the vector field F and fixes the diagonal Δ and thus could account for the existence of this invariant submanifold. Nor is there any F -preserving symmetry that fixes Δ' . In fact the vector field F doesn't seem to have any symmetry. The graph Γ recording the interdependence of the variables (x_1, x_2, x_3) in the ODE (1.1.1) has three vertices and three arrows:




The graph has no non-trivial symmetries. Nonetheless, the existence of the subsystems $(\Delta, F|_{\Delta})$, $(\Delta', F|_{\Delta'})$ and the whole diagram of the dynamical systems (1.1.2) can be deduced from certain properties of the graph Γ . There are two surjective maps of graphs:

$$\varphi : \Gamma \rightarrow \text{graph with one vertex and a self-loop}$$

¹The elements of category theory that are used in this paper are reviewed in Appendix A.

$$\psi : \Gamma \rightarrow \begin{array}{c} \circlearrowleft \\ a \end{array} \begin{array}{c} \circlearrowright \\ b \end{array},$$

Diagram illustrating the mapping of a graph structure. On the left, a graph with two nodes a and b connected by two curved edges (one top, one bottom) is labeled with a mapping τ . On the right, a graph with three nodes 1 , 2 , and 3 is shown. Nodes 1 and 2 are connected by two curved edges, and node 2 is connected to node 3 by a straight edge. An arrow points from the left graph to the right graph, indicating a mapping.

(1.1.3) 

$$\text{Loop on vertex} \xleftarrow{\varphi} \text{Vertex with incoming arrow} \xrightleftharpoons[\psi]{\tau} \text{Vertex with two outgoing arrows}$$

The same pattern holds when we replace the vector space \mathbb{R}^n by an arbitrary manifold M . Given a pair of manifolds U and N , we think of a map $X : U \times N \rightarrow TN$ with $X(u, n) \in T_n N$ as a control system with the points of U controlling the the dynamics on N . Now consider a vector field

of the form

for some control system

Then once again the three maps of graphs in the diagram (1.1.3) give rise to maps of dynamical systems

What accounts for the patterns we have seen? Notice that the dynamical systems (1.1.4) are constructed out of *one* control system $f : N \times N \rightarrow TN$. At the same time, in each of the graphs in (1.1.3), every vertex has exactly *one* incoming arc. This is not a coincidence. The rough idea for the technology which generalizes this example is this: if we have a dynamical system made up of repeated control system modules whose couplings are encoded in graphs, then the appropriate maps of graphs lift to maps of dynamical systems. Making this precise requires a number of constructions and theorems; these make up the bulk of this paper.

$$\mathbb{P} : \text{Graph}^{op} \rightarrow \text{Man}.$$

Associated to each node a of a directed graph Γ there is a set of edges of Γ with target a . These edges form the “input tree” $I(a)$ of a , which is itself a directed graph. With the help of the phase space functor \mathbb{P} , one can associate—to each such input tree $I(a)$ —a vector space $\text{Ctrl}(I(a))$ of control systems. These control systems are easy to describe; namely, the manifold assigned to

the leaves (which is itself a product of the individual manifolds associated to each leaf) controls the dynamics on the manifold assigned to the root.

The collection of input trees and their isomorphisms of a given graph Γ form a groupoid $G(\Gamma)$. This groupoid acts on the vector spaces of control systems attached to the input trees of the graph. We thus have a groupoid representation

$$\text{Ctrl}_\Gamma : G(\Gamma) \rightarrow \mathbf{Vect},$$

where \mathbf{Vect} is the category of (not necessarily finite-dimensional) real vector spaces and linear maps. It is natural to think of the limit of the functor Ctrl as the space $\mathbb{V}\Gamma$ of invariants of the representation. We think of $\mathbb{V}\Gamma$ as the collection of *virtual* groupoid-invariant vector fields on the phase space $\mathbb{P}(\Gamma)$. (We'll explain the meaning of the word "virtual" shortly.) It is not hard to check that the graphs in (1.1.3) produce the dynamical systems in (1.1.4). The story with the maps is a bit more complicated.

To extend the assignment $\Gamma \mapsto \mathbb{V}\Gamma$ to a functor we need to restrict ourselves to those morphisms of graphs that preserve the input trees. We call such maps of graphs *étale*². We have a subcategory $\mathbf{Graph}_{\text{et}}$ of the category of directed graphs and a contravariant functor

$$\mathbb{V} : (\mathbf{Graph}_{\text{et}})^{\text{op}} \rightarrow \mathbf{Vect}.$$

which extends the assignment $\Gamma \mapsto \mathbb{V}\Gamma$ to a contravariant functor on $\mathbf{Graph}_{\text{et}}$.

The elements of $\mathbb{V}(\Gamma)$ are not, strictly speaking, vector fields. But for each graph Γ there is a linear map

$$S = S_\Gamma : \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}(\Gamma)),$$

where $\chi(\mathbb{P}(\Gamma))$ denotes the space of vector fields on the manifold $\mathbb{P}(\Gamma)$. One may think of the image of S as the space of $G(\Gamma)$ invariant vector fields on $\mathbb{P}(\Gamma)$. In general the map S need not be injective. We are now in position to state the first result that explains the maps in (1.1.4):

Theorem. The functor \mathbb{V} and the collection of maps $\{S : \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}(\Gamma))\}$ are compatible: for any virtual groupoid invariant vector field $w \in \mathbb{V}(\Gamma')$ and any étale map of graphs $\varphi : \Gamma \rightarrow \Gamma'$ we have a map of dynamical systems

$$\mathbb{P}\varphi : (\mathbb{P}(\Gamma'), S(w)) \rightarrow (\mathbb{P}(\Gamma), S((\mathbb{V}(\varphi)w))).$$

In short, given any étale map between two colored graphs (e.g. the graph maps given in (1.1.3)), there is a gadget which intertwines all dynamical systems on the phase spaces associated to these graphs (the maps intertwining the dynamical systems in (1.1.4)).

Finally, the compatibility of \mathbb{V} and S can be expressed succinctly as follows. Let \mathcal{V} denote the category of elements of the functor \mathbb{V} . The objects of \mathcal{V} are pairs (Γ, w) where Γ is a graph and $w \in \mathbb{V}\Gamma$ is a virtual groupoid invariant vector field. We have a fibration of categories $\pi : \mathcal{V} \rightarrow (\mathbf{Graph}_{\text{et}})^{\text{op}}$. The assignment S extends to a functor $S : \mathcal{V} \rightarrow \mathbf{DynSys}$, and the diagram

$$(1.1.5) \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{S} & \mathbf{DynSys} \\ \pi \downarrow & & \downarrow U \\ (\mathbf{Graph}_{\text{et}})^{\text{op}} & \xrightarrow{\mathbb{P}} & \mathbf{Man} \end{array}$$

²Equivalently a map $\varphi : \Gamma \rightarrow \Gamma'$ of directed graphs is *étale* if for any vertex a of Γ and any edge e' of Γ' ending at $\varphi(a)$ there is a unique edge e of Γ ending at a with $\varphi(e) = e'$.

commutes. The objects of the category DynSys are pairs (M, X) consisting of a manifold M and a vector field X on M . The functor U is the functor that forgets the vector field: $U(M, X) = M$.

One final comment: the story above is not complete, since it treats all the nodes of the graphs, and all the arcs of the graphs, as being the same. In general one would want to consider the case where different vertices in the graph could correspond to different manifolds, and additionally, not all controls should be interchangeable. This is taken care of by passing to a relative version of the theory outlined above. More concretely, we fix a graph C of “colors”, choose an assignment of (possibly) different manifolds to different nodes of C , and an assignment of (possibly) different control systems to different edges of C . The category Graph_{et} is then replaced by the slice category $(\text{Graph}/C)_{\text{et}}$. We give a concrete example of the kind of dynamical system which requires such colors in Example 2.2.5.

1.2. Structure of paper. The paper is organized as follows. In Section 2 we develop the relevant mathematics under the assumptions that all of the manifolds are Euclidean (i.e. linear) spaces and the maps are smooth, i.e., the setting we consider are modular ODEs defined on copies of \mathbb{R}^n . This assumption allows for a direct comparison of our results with the results of Golubitsky et al. [49]. It will also be used in a subsequent paper on groupoid-compatible numerical methods. In Section 3 we specialize the results of Section 2 to the case where all the vector fields and maps are linear. We plan to use the results of this section in our subsequent work on stochastic systems. In Section 4 we generalize the theory of Section 2 to allow our phase spaces to be manifolds. In Section 5 we study groupoid invariant vector fields in the case where the underlying graph has nontrivial group symmetries and prove that the space of groupoid invariant vector fields is contained in the space of group-invariant vector fields. Section 6 contains several technical results about categorical limits. Finally in the Appendix we review the minimal amount of category theory that we need in this paper.

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2. GROUPOID INVARIANT VECTOR FIELDS ON EUCLIDEAN SPACES

2.1. Introduction.

In this section we present the theory sketched in the introduction in the setting of ordinary differential equations, that is, of vector fields on Euclidean spaces. As we mentioned in the introduction, we tackle Euclidean spaces first to ease a comparison of our results with those of Golubitsky et al. [49], and to allow us to build a setting for developing groupoid-compatible numerical methods among other reasons.

2.1.1. Any finite dimensional vector space V has a canonical structure of a second countable Hausdorff manifold. We will refer to this manifold as the *Euclidean space* V . Consequently it makes sense to talk about smooth vector fields on V . Moreover, there is a canonical trivialization of the tangent bundle TV of V :

$$V \times V \rightarrow TV, \quad (x, v) \mapsto \left. \frac{d}{dt} \right|_0 (x + tv),$$

which allows us to identify the space of smooth maps $C^\infty(V, V)$ from V to itself with the space $\chi(V)$ of vector fields on V . Explicitly

$$C^\infty(V, V) \ni f \mapsto X_f \in \chi(V), \quad X_f(v) = \left(v, \left. \frac{d}{dt} \right|_0 (v + tf(v)) \right).$$

If V and W are two *different* vector spaces, then a map from V to W is not (cannot be naturally identified with) a vector field on W , but it can be identified with a *control system* on W .

2.1.2. Definition. A *control system* on a manifold M is a pair $(p: Q \rightarrow M, F: Q \rightarrow TM)$, where

- (1) $p: Q \rightarrow M$ is a surjective submersion and
- (2) $F: Q \rightarrow TM$ is a smooth map with $F(q) \in T_{p(q)}M$ for all $q \in Q$.

(cf., for example, [55]). In particular, the following diagram

$$\begin{array}{ccc} Q & \xrightarrow{F} & TM \\ & \searrow p & \downarrow \pi \\ & & M \end{array}$$

commutes, where π is the canonical projection from the tangent bundle to its base.

2.1.3. Example. Let $Q = U \times M$ for some manifold U , and p be the projection onto the second factor. In this case, the control system consists of the *phase space* M and the *controls* U ; the function $F: Q = U \times M \rightarrow TM$ defines a nonautonomous ODE of the form

$$\dot{x} = F(u, x), \quad (u, x) \in U \times M.$$

2.1.4. Remark. Any vector field $X: M \rightarrow TM$ is a control system $(id: M \rightarrow M, X: M \rightarrow TM)$ — the system with *trivial* controls.

2.1.5. Remark. For a fixed surjective submersion $p: Q \rightarrow M$, the space

$$\text{CT}(p: Q \rightarrow M) := \{F: Q \rightarrow TM \mid F(q) \in T_{p(q)}M \text{ for all } q \in Q\}$$

of all smooth *control systems supported on* p is an infinite dimensional vector space.

2.1.6. Notation. We identify a smooth map $f: V \rightarrow W$ between two Euclidean spaces with the control system $(pr_2: V \times W \rightarrow W, F: V \times W \rightarrow TW)$ where

- (1) $pr_2: V \times W \rightarrow W$ is the projection on the second factor and
- (2) $F: V \times W \rightarrow TW$ is defined by

$$F(v, w) = (w, f(v)) \in T_w W \subset TW = W \times W.$$

In other words given two Euclidean spaces V and W we have a canonical embedding

$$(2.1.7) \quad \text{Hom}_{\text{Euc}}(V, W) \hookrightarrow \text{CT}(V \times W \xrightarrow{pr_2} W), \quad f \mapsto (pr_2, F)$$

where, as above, $\text{CT}(V \times W \xrightarrow{pr_2} W)$ denotes the space of all control systems supported by the projection $pr_2: V \times W \rightarrow W$, and $\text{Hom}_{\text{Euc}}(V, W) = C^\infty(V, W)$ is the space of infinitely differentiable maps from V to W .

2.1.8. Definition (Maps of control systems). Let $\{p_i: Q_i \rightarrow M_i, F_i: Q_i \rightarrow TM_i\}$, $i = 1, 2$, be a pair of control systems. A *morphism of control systems* from $\{p_1: Q_1 \rightarrow M_1, F_1: Q_1 \rightarrow TM_1\}$ to $\{p_2: Q_2 \rightarrow M_2, F_2: Q_2 \rightarrow TM_2\}$ is a pair of smooth maps $\varphi: Q_1 \rightarrow Q_2$, $\phi: M_1 \rightarrow M_2$ such that the two diagrams

$$\begin{array}{ccc} Q_1 & \xrightarrow{\varphi} & Q_2 \\ \downarrow p_1 & & \downarrow p_2 \\ M_1 & \xrightarrow{\phi} & M_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} Q_1 & \xrightarrow{\varphi} & Q_2 \\ \downarrow F_1 & & \downarrow F_2 \\ TM_1 & \xrightarrow{d\phi} & TM_2 \end{array}$$

commute.

Note that if the two control systems in question are vector fields $X_i: M_i \rightarrow TM_i$ (i.e. have trivial controls), then the morphism from the first to the second is a single map $\phi: M_1 \rightarrow M_2$ with

$$d\phi \circ X_1 = X_2 \circ \phi.$$

That is, X_1 and X_2 are ϕ -related.

Now suppose we have a smooth map $(X_1, \dots, X_n): V_1 \times \dots \times V_n \rightarrow V_1 \times \dots \times V_n$ from a product of Euclidean spaces V_1, \dots, V_n to itself thought of as a vector field. Then for each index j the pair

$$(p_j: V_1 \times \dots \times V_n \rightarrow V_j, \quad X_j: V_1 \times \dots \times V_n \rightarrow V_j)$$

is a control system. In other words the product $V_1 \times \dots \times V_n$ controls the dynamics on each factor V_j by way of the components X_j of the vector field X . Alternatively we can view $X = (X_1, \dots, X_n)$ as a collection of interacting control systems.

Suppose we know *a priori* that the dynamics on a factor V_j is controlled not by the full product $\prod V_i$ but by some subproduct $Q_j := V_{i_1} \times \dots \times V_{i_k}$ (the number k and the indices i_1, \dots, i_k depend on j). That is, suppose we can factor $X_j: \prod V_i \rightarrow V_j$ through the natural projection $\prod V_i \rightarrow V_{i_1} \times \dots \times V_{i_k}$. Then we can encode these facts—literally, which submodule controls which—by way of a *directed graph* (q.v. 2.2.1). Given a collection $X = (X_1, \dots, X_n)$ of interacting control systems on $V_1 \times \dots \times V_n$, we encode the dependence by a directed graph Γ in a natural way: the dynamics on the Euclidean space V_j depends only on the product $V_{i_1} \times \dots \times V_{i_k}$ if and only if the graph Γ has arrows $i_1 \rightarrow j, \dots, i_k \rightarrow j$.

Next suppose we know *a priori* that some control systems X_j above are “the same.” That is, the “control modules” are repeated through the system. Suppose further that some of the factors V_{i_ℓ} in the control/state Euclidean spaces $Q_j = V_{i_1} \times \dots \times V_{i_k}$ are repeated and that X_j ’s are invariant under the permutation of the repeated factors. What structure (in addition to the directed graph Γ) do we need to encode these assumptions? And what is the resulting spaces of these interacting control systems on $\prod V_i$ with the “symmetric modularity” described above? Our answer is a reworking and a substantial extension of ideas of Golubitsky et al. [46], [48].

The construction we are about to present requires a number of ingredients, which we develop throughout the rest of Section 2. The remainder of this section is broken up into two themes: in Subsections 2.2—2.8 we define the structure we study; in Subsections 2.9—2.12 we state and prove the main theorems justifying the construction and exhibiting the utility of this formalism. In more detail:

- (1) Construction of objects of study:
 - (a) Colored graphs, Subsection 2.2;
 - (b) Phase space functors \mathbb{P} , Subsection 2.3;
 - (c) The groupoid of colored trees $\mathbf{FinTree}/C$, Subsection 2.4;
 - (d) The control functor $\mathbf{Ctrl}: \mathbf{FinTree}/C \rightarrow \mathbf{Vect}$, Subsection 2.5;
 - (e) Input trees of a colored graph Γ forming a groupoid $G(\Gamma) \subset \mathbf{FinTree}/C$, Subsection 2.6;
 - (f) The space of groupoid-invariant vector fields $\mathbb{V}(\Gamma)$ as a limit of $\mathbf{Ctrl}|_{G(\Gamma)}$, Subsection 2.7;
 - (g) The relation of $\mathbb{V}(\Gamma)$ to vector fields on the phase space $\mathbb{P}\Gamma_0$, Subsection 2.8.
- (2) Results:
 - (a) Group-invariant versus groupoid-invariant vector fields* (this section may be skipped on the first reading), Subsection 2.9;
 - (b) The assignment $\Gamma \mapsto \mathbb{V}(\Gamma)$ extends to a contravariant functor $\mathbb{V}: (\mathbf{FinGraph}/C)_{et}^{\text{op}} \rightarrow \mathbf{Vect}$, Subsection 2.10;
 - (c) Virtual groupoid invariant vector fields and dynamics on Euclidean spaces, Subsection 2.11;
 - (d) The category of elements of the functor \mathbb{V} as a combinatorial category of groupoid invariant vector fields, Subsection 2.12.

Finally, we end the section with an extended example in Subsection 2.13. We now take them up one at a time.

2.2. Colored graphs.

We start by fixing the definition of a directed graph and of morphisms (maps) of directed graphs.

2.2.1. Definition. A *directed graph* (or, in this manuscript, *graph*) Γ consists of two sets Γ_1 (of arrows, or edges), Γ_0 (of nodes, or vertices) and two maps $s, t: \Gamma_1 \rightarrow \Gamma_0$ (source, target):

$$\Gamma = \{s, t: \Gamma_1 \rightarrow \Gamma_0\}.$$

We allow the possibility of Γ_1 being empty. We do not assume that Γ_0 is finite. We write $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$. A graph $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$ is *finite* if the sets Γ_0, Γ_1 of its nodes and edges are finite. A *map of graphs* $\varphi: \Gamma \rightarrow \Gamma'$ is a pair of maps of sets $\varphi_1: \Gamma_1 \rightarrow \Gamma'_1$ and $\varphi_0: \Gamma_0 \rightarrow \Gamma'_0$ so that the diagram

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varphi_1} & \Gamma'_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ \Gamma_0 \times \Gamma_0 & \xrightarrow{(\varphi_0, \varphi_0)} & \Gamma'_0 \times \Gamma'_0 \end{array}$$

commutes.

In order to keep track of various types of Euclidean spaces associated to the nodes and of the types of interactions we want our graph Γ “colored.” We note that the “colored” graphs defined below are not the standard colored graphs of the graph theory literature. Rather they are a variant of the colored graphs of Golubitsky et al. (op. cit.) who define a *coloring* of a graph $\Gamma = \{\Gamma_1 \rightrightarrows \Gamma_0\}$ to be a pair of compatible equivalence relations on the spaces Γ_1 of arrows and Γ_0 of nodes of the graph. But an equivalence relation \sim on a set X is the same thing as the quotient map $X \rightarrow X/\sim$. We find thinking in terms of maps rather than equivalence relations more natural, powerful and flexible. In particular it allows us to think of all compatibly colored graphs as a category. Thus our definition is:

2.2.2. Definition. Fix a directed graph C (“colors”). A *graph colored by C* is a map of graphs $\varphi: \Gamma \rightarrow C$. A *map of colored graphs* $f: (\varphi: \Gamma \rightarrow C) \rightarrow (\varphi': \Gamma' \rightarrow C)$ is a map of graphs $f: \Gamma \rightarrow \Gamma'$ with $\varphi' \circ f = \varphi$. That is, we have a commuting triangle:

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Gamma' \\ \varphi \searrow & & \swarrow \varphi' \\ & C & \end{array}$$

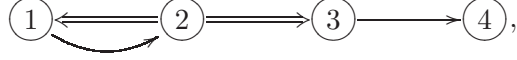
2.2.3. Remark. Given a node $x \in \Gamma$ we think of $\varphi(x) \in C_0$ as the color of x . Given an edge γ in Γ we think of $\varphi(\gamma)$ as the color of γ . Since φ is a map of graphs, the colors of edges and nodes are automatically compatible (q.v. Definition 2.2.1). Note that our colored graphs are colored graphs in the sense of Definition 5.1(f) in [49].

2.2.4. Remark. The category of graphs colored by a graph C is the slice category \mathbf{Graph}/C (see Definition A.1.21 in the Appendix). In the discussion that follows the color graph C is fixed. We often write Γ for an element of the slice category \mathbf{Graph}/C and $f: \Gamma \rightarrow \Gamma'$ for a morphism in \mathbf{Graph}/C (with all the maps to C suppressed in the notation).

2.2.5. Example. Imagine that we wanted to consider the vector field on $(\mathbb{R}^n)^4$ given by

$$x'_1 = f(x_2), \quad x'_2 = g(x_1), \quad x'_3 = f(x_2), \quad x'_4 = g(x_3),$$

where $f, g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are smooth functions. We want to encode the dependencies of different variables on one another, but also encode the fact that there are two types of control systems. We then encode these in the graph



so that the double arrow corresponds to the f control and the single arrow to the g control. One potentially misleading aspect of all of this is illustrated in the next example. If we now consider the vector fields on $(\mathbb{R}^n)^4$ encoded by the graph



then this corresponds to vector fields of the form

$$x'_1 = f(x_2), \quad x'_2 = g(x_1), \quad x'_3 = h(x_1, x_2), \quad x'_4 = g(x_3),$$

where we assume no relationship between the function h and the functions f, g . We find that this can be misleading, since intuitively the inputs to vertex 3 are given by two arrows which singly correspond to controls f and g , that the control from two vertices should somehow be related to f and g , but in the current formalism they are not. (We will consider formalisms the context of stochastic dynamical systems [54] where we assume some relationship between the controls encoded by collections of arrows and those encoded by the individual arrows in that collection, but not here.)

2.3. Phase space functors.

We need a consistent way of assigning phase spaces to nodes and to collections of nodes of various graphs colored by a graph $C = \{C_1 \rightrightarrows C_0\}$. Thus we need a functor from “colored sets” to Euclidean spaces. We construct such a functor \mathbb{P} in two steps. But first we need some notation.

2.3.1. Notation. The symbol \mathbf{FinSet} denotes the category of finite sets and maps of finite sets. The symbol \mathbf{FinSet}/C_0 denotes the slice category (q.v. Definition A.1.21) whose objects are maps of sets $\alpha: X \rightarrow C_0$, where C_0 is a fixed not necessarily finite set, X is a finite set and the morphisms are

$$\text{commuting triangles } \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \alpha & \swarrow \beta \\ & C_0 & \end{array} .$$

2.3.2. Notation. The symbol \mathbf{Euc} denotes the category of Euclidean spaces: the objects are finite dimensional vector spaces thought of as smooth manifolds (q.v. 2.1.1) and morphisms are infinitely differentiable (C^∞) maps.

Step 1. We choose a function $\mathcal{P}: C_0 \rightarrow \mathbf{Euc}_0$ from the set of colors of nodes C_0 to the collection \mathbf{Euc}_0 of objects of the category \mathbf{Euc} of Euclidean spaces. If we think of C_0 as a discrete category (q.v. Example A.1.8) then \mathcal{P} is a functor from C_0 to \mathbf{Euc} . We refer to \mathcal{P} as the *phase space function*.

Step 2. We use our choice of $\mathcal{P}: C_0 \rightarrow \mathbf{Euc}$ to define a contravariant phase space functor

$$(2.3.3) \quad \mathbb{P}: (\mathbf{FinSet}/C_0)^{op} \rightarrow \mathbf{Euc}$$

as follows. On objects of \mathbf{FinSet}/C_0 we set

$$\mathbb{P}X \equiv \mathbb{P}(X \xrightarrow{\alpha} C_0) := \prod_{x \in X} \mathcal{P}(\alpha(x)),$$

the *categorical product* of the family $\{\mathcal{P}(\alpha(x))\}_{x \in X}$ of the Euclidean spaces (q.v. Definition A.2.4). Note that since the set X is finite, the product $\prod_{x \in X} \mathcal{P}(\alpha(x))$ is a finite dimensional Euclidean

space and thus an object of the category **Euc** of Euclidean spaces. In particular if $X = \{1, 2, \dots, n\}$, then

$$\mathbb{P}X = \mathcal{P}(\alpha(1)) \times \dots \times \mathcal{P}(\alpha(n)),$$

the Cartesian product of n Euclidean spaces.

Given a morphism

$$(2.3.4) \quad \begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \beta & \swarrow \alpha \\ & C_0 & \end{array}$$

in the slice category \mathbf{FinSet}/C_0 , we construct $\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$ using the universal property of products: let $p_x: \mathbb{P}(X) \rightarrow \mathcal{P}(\alpha(x))$, $x \in X$, and $q_y: \mathbb{P}(Y) \rightarrow \mathcal{P}(\beta(y))$, $y \in Y$, denote the canonical projections. Then for any $y \in Y$ we have a map $p_{f(y)}: \mathbb{P}X \rightarrow \mathcal{P}(\alpha(f(y))) = \mathcal{P}(\beta(y))$. Therefore, by the universal property of product $\mathbb{P}Y$ (q.v. A.2.4) there is a unique map $\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$ making the diagram

$$(2.3.5) \quad \begin{array}{ccc} \mathbb{P}X & \xrightarrow{\quad \mathbb{P}f \quad} & \mathbb{P}Y \\ p_{f(y)} \downarrow & & \downarrow q_y \\ \mathcal{P}(\alpha(f(y))) & \xlongequal{\quad} & \mathcal{P}(\beta(y)) \end{array}$$

commute.

2.3.6. Example. Suppose C_0 is a the two element set $\{c_1, c_2\}$, $\mathcal{P}(c_1)$ is a Euclidean space V , $\mathcal{P}(c_2)$ is a Euclidean space W , $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $\alpha(x_i) = c_i$, $i = 1, 2$, $\beta(y_i) = c_1$ for all i and $f(y_1) = f(y_2) = x_1$. Then

$$\mathbb{P}X = \mathcal{P}(\alpha(x_1)) \times \mathcal{P}(\alpha(x_2)) = V \times W,$$

$$\mathbb{P}Y = \mathcal{P}(\beta(y_1)) \times \mathcal{P}(\beta(y_2)) = V \times V,$$

$p_{f(y_1)} = p_{f(y_2)} = p_{x_1}$, which is the canonical projection $V \times W \rightarrow V$, while $q_{y_1}: V \times V \rightarrow V$ is the projection on the first factor and $q_{y_2}: V \times V \rightarrow V$ is the projection on the second. Consequently $\mathbb{P}f: V \times W \rightarrow V \times V$ is the unique map with $q_{y_1} \circ \mathbb{P}f(v, w) = v$ and $q_{y_2} \circ \mathbb{P}f(v, w) = v$. That is

$$\mathbb{P}f(v, w) = (v, v).$$

2.3.7. Example. More generally, suppose C_0 is the set of natural numbers \mathbb{N} , $X = \{1, \dots, N\}$, $Y = \{1, \dots, M\}$, and $\alpha: X \rightarrow C_0$, $\beta: Y \rightarrow C_0$ and $f: Y \rightarrow X$ are three maps with $\alpha \circ f = \beta$. Then for any sequence $\{\mathcal{P}(i)\}_{i \in \mathbb{N}}$ of Euclidean spaces we have

$$\mathbb{P}Y = \mathcal{P}(\beta(1)) \times \dots \times \mathcal{P}(\beta(M)), \quad \mathbb{P}X = \mathcal{P}(\alpha(1)) \times \dots \times \mathcal{P}(\alpha(N)),$$

$q_j: \mathbb{P}Y \rightarrow \mathcal{P}(\beta(j))$ is given by

$$q_j(w_1, \dots, w_M) = w_j,$$

$p_i: \mathbb{P}X \rightarrow \mathcal{P}(\alpha(i))$ is given by

$$p_i(v_1, \dots, v_N) = v_i,$$

and $\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$ is defined by (2.3.5), that is by

$$q_j(\mathbb{P}f(v)) = p_{f(j)}(v) = v_{f(j)}.$$

Hence

$$(2.3.8) \quad \mathbb{P}f(v_1, \dots, v_N) = (v_{f(1)}, \dots, v_{f(N)}).$$

- 2.3.9. Remark.** (1) Suppose $X \xrightarrow{\alpha} C_0, Y \xrightarrow{\beta} C_0 \in \mathbf{FinSet}/C_0$ are singletons: $X = \{x\}, Y = \{y\}$. Then any morphism $f: Y \rightarrow X$ in \mathbf{FinSet}/C_0 has to map the unique element y of Y to the unique element x of X and we must have $\alpha(f(y)) = \beta(y)$. Then $\mathbb{P}X = \mathcal{P}(\alpha(x)) = \mathcal{P}(\beta(y)) = \mathbb{P}Y$ and $\mathbb{P}f$ is forced to be the identity map $id_{\mathbb{P}X}$.
- (2) If $X \in \mathbf{FinSet}/C_0$ is empty then $\mathbb{P}X$ is the one point Euclidean space $\{0\}$ (q.v. A.2.6).
- (3) It is not hard to check that if $X = Y$ and f is the identity map id_X then $\mathbb{P}(id_X) = id_{\mathbb{P}X}$ (note that if f is id_X then $\alpha = \beta$). Also, using the universal property of categorical products it is easy to check that if

$$\begin{array}{ccccc} Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \\ & \searrow \gamma & \downarrow \beta & \swarrow \alpha & \\ & & C & & \end{array}$$

is a pair of composable morphisms in \mathbf{FinSet}/C_0 , then

$$\mathbb{P}(f \circ g) = \mathbb{P}g \circ \mathbb{P}f.$$

In other words \mathbb{P} is a contravariant functor

$$(2.3.10) \quad \mathbb{P}: (\mathbf{FinSet}/C_0)^{\text{op}} \rightarrow \mathbf{Euc}.$$

2.3.11. Remark. It follows from Example 2.3.7 (in particular from (2.3.8)) that for any map $f: Y \rightarrow X$ of finite sets over C_0 the map

$$\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$$

of Euclidean spaces is actually *linear*.

2.3.12. Remark. The only property of the category \mathbf{Euc} of Euclidean spaces that we used in construction the functor \mathbb{P} is that \mathbf{Euc} has finite products. Therefore the same construction works for the categories $\mathbf{FinVect}$ of finite dimensional vector spaces and linear maps and \mathbf{Man} of manifolds. Thus given a choice of a functor $\mathcal{P}: C_0 \rightarrow \mathbf{FinVect}$ (again we think of C_0 as a discrete category) we get a contravariant functor

$$\mathbb{P}: (\mathbf{FinSet}/C_0)^{\text{op}} \rightarrow \mathbf{FinVect};$$

given a choice of a functor $\mathcal{P}: C_0 \rightarrow \mathbf{Man}$ we get a a contravariant functor

$$\mathbb{P}: (\mathbf{FinSet}/C_0)^{\text{op}} \rightarrow \mathbf{Man}.$$

We will use these maps in Sections 3 and 4, respectively.

2.3.13. Notation. Fix a directed graph $C = \{C_1 \rightrightarrows C_0\}$. The symbol $\mathbf{FinGraph}/C$ denotes the full subcategory of the slice category \mathbf{Graph}/C whose objects are $(f: \Gamma \rightarrow C) \in (\mathbf{Graph}/C)_0$ with Γ a *finite* graph.

2.3.14. Remark. There is an evident forgetful functor

$$F: \mathbf{FinGraph}/C \rightarrow \mathbf{FinSet}/C_0, \quad F(\Gamma \rightarrow C) := (\Gamma_0 \rightarrow C_0),$$

which forgets the arrows of a graph $\Gamma \rightarrow C$. Following F by the phase space functor \mathbb{P} gives us a contravariant functor $(\mathbf{FinGraph}/C)^{\text{op}} \rightarrow \mathbf{Euc}$. We will abuse the notation and denote the composite $\mathbb{P} \circ F$ by \mathbb{P} . Thus

$$\mathbb{P}(\Gamma \xleftarrow{h} \Gamma') = \mathbb{P}\Gamma \xrightarrow{\mathbb{P}h} \mathbb{P}\Gamma' \equiv \mathbb{P}\Gamma_0 \xrightarrow{\mathbb{P}h} \mathbb{P}\Gamma'_0$$

for any morphism $h: \Gamma' \rightarrow \Gamma$ in $\mathbf{FinGraph}/C$.

2.4. The groupoid of finite colored trees $\text{FinTree}/C$.

2.4.1. Definition. A *path* in a directed graph Γ is a sequence of edges with the matching sources and targets:

$$a_0 \xleftarrow{\varepsilon^1} a_1 \xleftarrow{\varepsilon^2} \dots \xleftarrow{\varepsilon^n} a_n.$$

That is, $a_i \in \Gamma_0$ are vertices, $a_{i-1} \xleftarrow{\varepsilon^i} a_i \in \Gamma_1$ are edges and, additionally, $s(e_i) = a_i = t(e_{i+1})$. Paths of length zero are by definition vertices of Γ .

2.4.2. Definition (Finite tree). A finite directed graph T is a *tree* if there is exactly one path between any two distinct vertices of T . The root of a tree T is the unique vertex with no outgoing edges. A vertex of T that has no incoming edges and which is not a root is called a *leaf* of T . We denote the singleton set consisting of the root of T by $\text{rt } T$. We denote the set of leaves of a tree T by $\text{lv } T$.

2.4.3. Remark. Note that by our definition a graph T consisting of a single vertex a and no edges (that is, $T = \{\emptyset \rightrightarrows \{a\}\}$) is a tree. The root of this tree is the vertex a and the set of leaves is empty.

2.4.4. Definition (The category of finite colored trees). Fix a graph C of colors. The category $\text{FinTree}/C$ of *finite colored trees over C* is defined by

$$(\text{FinTree}/C)_0 = \text{finite trees colored by } C = \{T \rightarrow C \mid T \text{ finite tree}\},$$

$$\text{Hom}_{\text{FinTree}/C}(T, T') = \left\{ \begin{array}{c} T \xrightarrow{\sigma} T' \\ \alpha \searrow \swarrow \beta \\ C \end{array} \middle| \sigma \text{ is an isomorphism of graphs over } C \right\}.$$

Note that by definition the category $\text{FinTree}/C$ is a groupoid (q.v. Definition A.1.16).

2.5. Control systems functor.

2.5.1. Definition (The space of control systems associated to a tree). Once again we fix a graph C and a function $\mathcal{P}: C_0 \rightarrow \text{Euc}$. Let $T \rightarrow C$ be a finite tree over C . We define the infinite-dimensional vector space of *control systems associated to the tree $T \rightarrow C$* by

$$\text{Ctrl}(T) := \text{Hom}_{\text{Euc}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T) \equiv C^\infty(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T).$$

That is, $\text{Ctrl}(T)$ is the space of all smooth maps from the Euclidean space $\mathbb{P} \text{lv } T$ to the Euclidean space $\text{rt } T$.

- 2.5.2. Remark.**
- (1) By the identification in Notation 2.1.6, each map $f \in \text{Hom}_{\text{Euc}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T)$ defines a control system $(\mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow \mathbb{P} \text{rt } T, \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow T \mathbb{P} \text{rt } T)$
 - (2) If the space of leaves $\text{lv } T$ is empty (so that T consists of a single vertex) then $\mathbb{P} \text{lv } T = \{0\}$ and $\text{Ctrl}(T) = \text{Hom}_{\text{Euc}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T) = \text{Hom}_{\text{Euc}}(\{0\}, \mathbb{P} \text{rt } T) = \{0\}$, the zero dimensional vector space.
 - (3) Our definition only considers the top and bottom generations of the tree and ignores everything in the middle. We will in fact only consider one-generation trees below, so this restriction will not be important.

The assignment $T \mapsto \text{Ctrl}(T)$ extends to a functor $\text{Ctrl}: \text{FinTree}/C \rightarrow \text{Vect}$, from the category of

finite trees over C to the category of infinite dimensional vector spaces as follows. Let $\begin{array}{c} T \xrightarrow{\sigma} T' \\ \alpha \searrow \swarrow \beta \\ C \end{array}$

be an isomorphism of trees. Then we have two isomorphisms of finite sets over C_0 :

$$\sigma|_{\text{lv } T}: \text{lv } T \rightarrow \text{lv } T' \quad \text{and} \quad \sigma|_{\text{rt } T}: \text{rt } T \rightarrow \text{rt } T'.$$

Applying the phase space functor \mathbb{P} we get two diffeomorphisms:

$$\mathbb{P}(\sigma|_{\text{lv } T}): \mathbb{P} \text{lv } T' \rightarrow \mathbb{P} \text{lv } T, \quad \mathbb{P}(\sigma|_{\text{rt } T}): \mathbb{P} \text{rt } T' \rightarrow \mathbb{P} \text{rt } T.$$

Note that the second diffeomorphism is the identity map by Remark 2.3.9. Consequently we obtain the map

$$(2.5.3) \quad \text{Ctrl}(\sigma): \text{Ctrl}(T) \rightarrow \text{Ctrl}(T')$$

given by

$$(2.5.4) \quad \text{Ctrl}(\sigma)X: = X \circ \mathbb{P}(\sigma|_{\text{lv } T}): \mathbb{P} \text{lv } T' \rightarrow \mathbb{P} \text{rt } T = \mathbb{P} \text{rt } T'$$

for any $(X: \mathbb{P} \text{lv } T \rightarrow \mathbb{P} \text{rt } T) \in \text{Ctrl}(T) = \text{Hom}_{\text{Euc}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T)$, so that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P} \text{lv } T & \xrightarrow{X} & \mathbb{P} \text{rt } T \\ \uparrow \mathbb{P}(\sigma|_{\text{lv } T}) & & \parallel \mathbb{P}(\sigma|_{\text{rt } T}) = \text{id} \\ \mathbb{P} \text{lv } T' & \xrightarrow{\text{Ctrl}(\sigma)X} & \mathbb{P} \text{rt } T' \end{array}$$

We leave it to the reader to check that Ctrl preserves the composition of morphisms and thus a functor.

2.6. Input trees of colored graphs.

2.6.1. Definition (Input tree). Given a vertex a of a graph Γ we define the *input tree* $I(a)$ to be the following graph:

- the vertices $I(a)_0$ of $I(a)$ is the set

$$I(a)_0 := \{a\} \sqcup \coprod t^{-1}(a),$$

where $t^{-1}(a)$ is the set of arrows in Γ with target a ;

- the edges $I(a)_1$ of $I(a)$ is the set of pairs

$$I(a)_1 := \{(a, \gamma) \mid t(\gamma) = a\},$$

- the source and target maps $I(a)_1 \rightrightarrows I(a)_0$ are defined by

$$s(a, \gamma) = \gamma \quad \text{and} \quad t(a, \gamma) = a.$$

In pictures

$$\gamma \bullet \xrightarrow{(a, \gamma)} \bullet a.$$

It is easy to see that the graph $I(a)$ is a tree with

$$\text{rt } I(a) = \{a\}, \quad \text{lv } I(a) = t^{-1}(a)$$

2.6.2. Remark. We have a natural map of graphs $\xi: I(a) \rightarrow \Gamma$ given on arrows by $\xi(a, \gamma) = \gamma$. Thus if $\Gamma \rightarrow C$ is a graph over C then so are its input trees $I(a)$ for each vertex $a \in \Gamma_0$, and the composite $I(a) \xrightarrow{\xi} \Gamma \xrightarrow{c} C$ makes $I(a)$ into a graph over C .

2.6.3. Definition (The groupoid $G(\Gamma)$). Given a graph $\Gamma \rightarrow C$ over C we define its *symmetry groupoid* $G(\Gamma)$ to be the groupoid with the set of objects $G(\Gamma)_0$ equal to the set of the input trees of Γ :

$$G(\Gamma)_0 = \{I(a) \mid a \in \Gamma_0\};$$

and the sets of morphisms

$$\text{Hom}_{G(\Gamma)}(I(a), I(b)) := \{\sigma: I(a) \rightarrow I(b) \mid \sigma \text{ is an isomorphism of graphs over } C\}.$$

In other words $G(\Gamma)$ is the full subcategory of $\text{FinTree}/C$ with the set of objects $\{I(a)\}_{a \in \Gamma_0}$.

2.7. Virtual groupoid invariant vector fields.

Given the groupoid of input trees $G(\Gamma)$ of a graph Γ we restrict the functor Ctrl to $G(\Gamma)$ and get a functor

$$\text{Ctrl}_\Gamma = \text{Ctrl}|_{G(\Gamma)}: G(\Gamma) \rightarrow \text{Vect}.$$

2.7.1. Remark. For an input tree $I(a)$ of a graph $\Gamma \rightarrow C$ the space $\text{Ctrl}(I(a))$ is the space of all smooth maps from the phase space $\mathbb{P}\text{lv } I(a)$ of leaves to the phase space of the root $\mathbb{P}\{a\}$. Chasing through the definitions we see that $\mathbb{P}\text{lv } I(a)$ is a finite product of Euclidean spaces, one Euclidean space for each *arrow* pointing into the node a . If all the arrows with target a have distinct sources, then $\mathbb{P}\text{lv } I(a)$ is the same as the product of the appropriate vector spaces indexed by these sources, which is the definition of Golubitsky–Pivato–Stewart in [46]. However, note that if there is a node b of Γ which is connected to a by multiple arrows then our definition is different from the Golubitsky–Stewart–Török definition [48]. The difference can be seen in the Example 2.7.4 below.

2.7.2. Definition (Virtual groupoid invariant vector fields). Fix the graph of colors $C = \{C_1 \rightrightarrows C_0\}$ and a phase space function $\mathcal{P}: C_0 \rightarrow \text{Euc}$. Given a graph Γ over C we define the vector space of *virtual groupoid invariant vector fields* $\mathbb{V}(\Gamma)$ to be the limit of the functor Ctrl_Γ (q.v. Definition A.2.12 and the subsequent discussion):

$$\mathbb{V}(\Gamma) := \lim(\text{Ctrl}_\Gamma: G(\Gamma) \rightarrow \text{Vect}).$$

Note that since $\mathbb{V}(\Gamma)$ is a limit, it comes with a family of the canonical projections

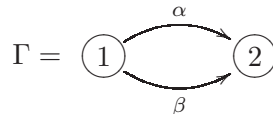
$$\{\varpi_a: \mathbb{V}(\Gamma) \rightarrow \text{Ctrl}(I(a))\}_{a \in \Gamma_0}.$$

2.7.3. Remark. The vector space $\mathbb{V}(\Gamma)$ has the following concrete description. Consider the product $\prod_{a \in \Gamma_0} \text{Ctrl}(I(a))$ with its canonical projections $p_b: \prod_{a \in \Gamma_0} \text{Ctrl}(I(a)) \rightarrow \text{Ctrl}(I(b))$, $b \in \Gamma_0$. For a vector $X \in \prod_{a \in \Gamma_0} \text{Ctrl}(I(a))$ we denote the a -th component $p_a(X)$ of X by X_a . Then

$$\mathbb{V}(\Gamma) = \left\{ X \in \prod_{a \in \Gamma_0} \text{Ctrl}(I(a)) \mid \text{Ctrl}(\sigma)X_a = X_b \text{ for all arrows } I(a) \xrightarrow{\sigma} I(b) \text{ in the groupoid } G(\Gamma) \right\}$$

with the canonical projections $\varpi_a: \mathbb{V}(\Gamma) \rightarrow \text{Ctrl}(I(a))$ given by restrictions $p_a|_{\mathbb{V}(\Gamma)}$. To check that the collection $\{\varpi_a: \mathbb{V}(\Gamma) \rightarrow \text{Ctrl}(I(a))\}_{a \in \Gamma_0}$ is in fact a limit of Ctrl_Γ it is enough to check its universal properties, which we leave to the reader.

2.7.4. Example. Let C be a graph with one vertex v and one edge: $C = \circ \curvearrowright \circ$. Then any graph Γ is canonically a graph over C and the category of finite graphs over C is isomorphic to the category of finite graphs. Let $\mathcal{P}: C_0 = \{\circ\} \rightarrow \text{Euc}$ be given by $\mathcal{P}(\circ) = V$ for some Euclidean space V . Let Γ be the graph with two nodes and two arrows:



We have $\Gamma_0 = \{1, 2\}$ and $\mathbb{P}\Gamma_0 = V \times V$. The input trees of the two nodes are

$$I(1) = \textcircled{1}, \quad I(2) = \begin{array}{c} \{\alpha\} \xrightarrow{\alpha} \\ \{\beta\} \xrightarrow{\beta} \end{array} \textcircled{2}.$$

Since $\text{lv}(I(1)) = \emptyset$, we have $\text{Ctrl}(I(1)) = \{0\}$. The set of leaves of the input tree $I(2)$ has two elements: $\text{lv}(I(2)) = \{\alpha, \beta\}$. Through the map ξ (q.v. Remark 2.6.2) the leaves of $I(2)$ inherit the colors of their sources. Hence $\mathbb{P}\text{lv}(2) = V \times V$. Therefore $\text{Ctrl}(I(2)) = \text{Hom}_{\text{Euc}}(V \times V, V) = C^\infty(V \times V, V)$.

The groupoid $G(\Gamma)$ has two objects: $I(1)$ and $I(2)$. The only nontrivial morphism of $G(\Gamma)$ is the map $\sigma : I(2) \rightarrow I(2)$ which exchanges α and β :

$$G(\Gamma) : \quad I(1) \quad I(2) \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \sigma$$

Consequently

$$(\text{Ctrl}(\sigma)f)(v, w) = f(w, v)$$

for any $f \in \text{Ctrl}(I(2))$ and any $(v, w) \in \mathbb{P}\text{lv} I(2)$. We conclude that

$$\mathbb{V}(\Gamma) = \{0\} \oplus C^\infty(V \times V, V)^{\mathbb{Z}_2} \simeq C^\infty(V \times V, V)^{\mathbb{Z}_2},$$

where \mathbb{Z}_2 denotes the two-element group. Now consider the graph Γ' :

$$\Gamma' = \begin{array}{c} \textcircled{1a} \xrightarrow{\alpha} \\ \textcircled{1b} \xrightarrow{\beta} \end{array} \textcircled{2}$$

The groupoid $G(\Gamma')$ has three objects: the trees $I(1a)$, $I(1b)$ and $I(2)$. It has three nontrivial arrows:

$$G(\Gamma') = \begin{array}{c} I(1a) \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\ I(1b) \end{array} \quad I(2) \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \sigma$$

A calculation similar to the one above shows that

$$\mathbb{V}(\Gamma') = \{0\} \oplus \{0\} \oplus C^\infty(V \times V, V)^{\mathbb{Z}_2}.$$

2.8. The relation of $\mathbb{V}(\Gamma)$ to vector fields on the phase space $\mathbb{P}\Gamma_0$.

2.8.1. Recall that the space of vector fields $\chi(W)$ on a Euclidean space W is canonically isomorphic to $\text{Hom}_{\text{Euc}}(W, W) \equiv C^\infty(W, W)$ (q.v. 2.1.1).

Given a finite graph Γ over a graph of colors C and the phase space function $\mathcal{P} : C_0 \rightarrow \text{Euc}$, the functor \mathbb{P} assigns a phase space $\mathbb{P}\Gamma_0$ to the set Γ_0 of vertices of Γ . There exists a canonical *linear* map S_Γ from the space of virtual groupoid-invariant vector fields $\mathbb{V}(\Gamma)$ to the vector space $\chi(\mathbb{P}\Gamma_0)$ of vector fields on $\mathbb{P}\Gamma_0$, which we now define.

Since $\chi(\mathbb{P}\Gamma_0) = \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0) = \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \prod_{a \in \Gamma_0} \mathbb{P}\{a\})$ (q.v. Remark A.2.8), the space of vector fields on $\mathbb{P}\Gamma_0$ is canonically the product

$$\chi(\mathbb{P}\Gamma_0) = \prod_{a \in \Gamma_0} \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}).$$

To define the map into a product, it is enough to define a map into its factors. Thus we need maps $\mathbb{V}(\Gamma) \rightarrow \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\})$ for each vertex a of the graph Γ .

For each input tree $I(a)$ we have a map of graphs $\xi: I(a) \rightarrow \Gamma$ (q.v. Remark 2.6.2). Therefore we have a map $\xi|_{\text{lv } I(a)}: \text{lv } I(a) \rightarrow \Gamma_0$ of finite sets over C_0 . Applying the phase space functor \mathbb{P} we get maps

$$\mathbb{P}(\xi|_{\text{lv } I(a)}): \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\text{lv } I(a).$$

The pullback by $\mathbb{P}(\xi|_{\text{lv } I(a)})$ gives

$$(\mathbb{P}(\xi|_{\text{lv } I(a)}))^*: \text{Ctrl}(I(a)) \rightarrow \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}), \quad F \mapsto F \circ \mathbb{P}(\xi|_{\text{lv } I(a)}).$$

By the universal properties of the product $\chi(\mathbb{P}\Gamma_0)$ there a unique canonical map $S_\Gamma: \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0)$ making the diagram

$$(2.8.2) \quad \begin{array}{ccc} \mathbb{V}(\Gamma) & \overset{\exists! S_\Gamma}{\dashrightarrow} & \chi(\mathbb{P}\Gamma_0) \\ \varpi_a \downarrow & & \downarrow p_a \\ \text{Ctrl}(I(a)) & \xrightarrow{(\mathbb{P}(\xi|_{\text{lv } I(a)}))^*} & \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}) \end{array}$$

commute (see Definition A.2.1). Here, as before,

$$(2.8.3) \quad \varpi_a: \mathbb{V}\Gamma \rightarrow \text{Ctrl}(I(a))$$

and

$$(2.8.4) \quad p_a: \chi(\mathbb{P}\Gamma_0) = \prod_{a' \in \Gamma_0} \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a'\}) \rightarrow \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\})$$

denote the canonical projections.

2.8.5. Example. We return to the graphs Γ and Γ' of Example 2.7.4 and the phase space function $\mathcal{P}(\circ) = V$. The space $\chi(\mathbb{P}\Gamma_0)$ of vector fields on $\mathbb{P}\Gamma_0$ is $C^\infty(V^2, V^2)$ and $\chi(\mathbb{P}\Gamma'_0) = C^\infty(V^3, V^3)$. Unraveling the definitions we see that

$$S_\Gamma: \mathbb{V}(\Gamma) = C^\infty(V \times V, V)^{\mathbb{Z}_2} \rightarrow C^\infty(V^2, V^2) = \chi(\mathbb{P}\Gamma_0)$$

is given by

$$(S_\Gamma(f))(v_1, v_2) = (0, f(v_1, v_1))$$

for all $(v_1, v_2) \in V \times V$. This is because $\xi|_{\text{lv } I(2)}$ is the map sending both nodes of $\text{lv } I(2)$ to $\textcircled{1}$. In the case of Γ'

$$S_{\Gamma'}: \mathbb{V}(\Gamma') = C^\infty(V \times V, V)^{\mathbb{Z}_2} \rightarrow C^\infty(V^3, V^3) = \chi(\mathbb{P}\Gamma'_0)$$

is given by

$$(S_{\Gamma'}(f))(v_{1a}, v_{1b}, v_2) = (0, 0, f(v_{1a}, v_{1b})).$$

Note that $S_{\Gamma'}$ is injective, while S_Γ is not: S_Γ vanishes on all the functions f whose restriction to the diagonal $\Delta_V \subset V \times V$ is zero.

2.9. Group-invariant versus groupoid-invariant vector fields. ³

In laying out the main themes of the survey article [49] Golubitsky and Stewart list a number of questions “raised by the groupoid point of view.” In particular they ask (op. cit., p. 309):

“What are the analogies with the group case? When do these analogies fail and why?”

In this subsection we provide a partial answer and suggest an approach to answering this question more fully. In the previous subsections we constructed the space of virtual groupoid-invariant vector fields as “fixed points” of a groupoid representation. Just as in the case of groups, there is a notion of a groupoid action that is more general than that of a groupoid representation. To recall the relevant definition we need more notation.

2.9.1. Notation. Given maps of sets $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ the *fiber product* $X \times_{f,Z,g} Y$ is the set

$$X \times_{f,Z,g} Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\}.$$

(Note that the fiber product of f and g together with the canonical projections $p_X: X \times_{f,Z,g} Y \rightarrow X$, $p_Y: X \times_{f,Z,g} Y \rightarrow Y$ is the limit of a functor from the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ with 3 elements and 2 non-identity maps to the category **Set** of sets which sends the non-identity arrows to f and g respectively; q.v. A.2.1.)

2.9.2. Definition. A *action* of a groupoid $G = \{G_1 \rightrightarrows G_0\}$ on a set X is a pair of maps $\mathbf{an}: X \rightarrow G_0$ (anchor) and $\mathbf{a}: G_1 \times_{s,G_0,\mathbf{an}} X \rightarrow X$ (action; $s: G_1 \rightarrow G_0$ denotes the source map) so that

- (1) $\mathbf{an}(\mathbf{a}(g, x)) = t(g)$ for all $(g, x) \in G_1 \times_{s,G_0,\mathbf{an}} X$,
- (2) $\mathbf{a}(1_{\mathbf{an}(x)}, x) = x$ for all $x \in X$ and
- (3) $\mathbf{a}(g_2 g_1, x) = \mathbf{a}(g_2, \mathbf{a}(g_1, x))$ for all composable arrows $(g_2, g_1) \in G_1 \times_{s,G_0,t} G_1$ ($t: G_1 \rightarrow G_0$ is the target map) and all $x \in X$ with $\mathbf{an}(x) = s(g_1)$.

It is not hard to see that a functor $\rho: G \rightarrow \mathbf{Vect}$ from a (small) groupoid G to the category **Vect** of vector spaces defines an action of G on the set $X = \bigsqcup_{c \in G_0} \rho(c)$. The anchor $\mathbf{an}: X \rightarrow G_0$ is defined by $\mathbf{an}(\rho(c)) = c$. The action \mathbf{a} is given by

$$\mathbf{a}(g, v) = \rho(g)v.$$

There is, however, one crucial difference between *groupoid*-invariant vector fields and *group*-invariant vector fields as they are studied in, for example, [30] or [56]. In the case of the group-invariant vector fields on a phase space M (M is a Euclidean space or, more generally, a manifold) the action of a group H on the space of vector fields is *induced by an action of H on M* :

$$(h \cdot v)(m) := (Dh_M)_m(v(h_M^{-1}(m)))$$

for all group elements $h \in H$, points $m \in M$ and vector fields $v \in \chi(M)$. Here $h_M: M \rightarrow M$ denotes the diffeomorphism defined by the action of h on M :

$$h_M(m) = h \cdot m.$$

In the case of groupoid-invariant vector fields (q.v. 2.7.2) the action of a groupoid $G(\Gamma)$ on the set of the associated control systems $\bigsqcup_{a \in \Gamma_0} \mathbf{Ctrl}(I(a))$ is *not induced by any action of the groupoid $G(\Gamma)$ on the phase space $\mathbb{P}\Gamma_0$* . Consequently, the intuition acquired in the studies of group-invariant dynamical systems can be as much of a hindrance as of help in thinking about groupoid invariant dynamical systems.

We speculate that to fully answer the question of Golubitsky and Stewart quoted above one would need, on one hand, to understand dynamical systems with group-invariant vector fields where the

³This subsection may be skipped on the first reading

group action *in not induced* by an action of the group on the phase space, and, on the other hand, to study groupoid invariant vector fields where the action of the groupoid on the vector fields *is induced* by the action of the groupoid on the phase space.

2.10. The assignment $\Gamma \mapsto \mathbb{V}\Gamma$ extends to a functor.

A map $\varphi: \Gamma \rightarrow \Gamma'$ of graphs over a fixed graph C induces, for each node $a \in \Gamma_0$ a map of input trees

$$\varphi_a: I(a) \rightarrow I(\varphi(a)),$$

on arrows

$$(2.10.1) \quad \varphi_a(a, \gamma) = (\varphi(a), \varphi(\gamma)).$$

However, it does not, in general, induce a map from the groupoid $G(\Gamma)$ to the groupoid $G(\Gamma')$ let alone a map between the spaces of virtual invariant vector fields $\mathbb{V}(\Gamma)$ and $\mathbb{V}(\Gamma')$.

2.10.2. Definition. A map $\varphi: \Gamma \rightarrow \Gamma'$ of graphs over a graph C of colors is *étale* (or a *local isomorphism*) if, for each node a of the graph Γ , the induced map

$$\varphi_a: I(a) \rightarrow I(\varphi(a))$$

of input trees defined by (2.10.1) is an isomorphism of graphs over C .

Note that the outgoing edges play no role in the definition of an étale map.

2.10.3. Remark. The notion of an étale map of directed graphs is not new and goes by many different names [57,58]. Calling it a “local isomorphism” could be misleading, since we only consider incoming arrows, and it might better be termed “local in-isomorphism” as in [57,58]. In Higgins [59] it is called a covering map, but since local isomorphisms of graphs are not necessary surjective on nodes, calling it a “covering map” may give a wrong impression. Calling them “étale” is a lot shorter than calling them “local isomorphisms.” See [60] for an extensive discussion of the history of the notion and various contexts in which it arouse.

2.10.4. Remark. If a morphism $\varphi: \Gamma \rightarrow \Gamma'$ in \mathbf{Graph}/C is étale than for each node a of Γ and each edge γ' in Γ' with $t(\gamma') = \varphi(a)$ there exists a unique edge γ in Γ with $\varphi(\gamma) = \gamma'$. Consequently given any path in Γ' with the end node $f(a)$ there is a unique lift of this path to a path in Γ with the end node a (q.v. footnote 2 on p. 4). We will not use the unique lifting property of paths in this paper, but it will play an important role in the subsequent paper on the groupoid-invariant discrete-time dynamics on networks.

2.10.5. Remark. Finite graphs over a fixed graph C , along with étale maps, form a subcategory of the category $\mathbf{FinGraph}/C$ of finite graphs over C , since the composition of étale maps is étale. We call this subcategory $(\mathbf{FinGraph}/C)_{\text{et}}$.

2.10.6. Theorem. *The map \mathbb{V} that assigns to each finite graph Γ over C the vector space of $G(\Gamma)$ -invariant virtual vector fields on $\mathbb{P}\Gamma_0$ extends to a contravariant functor*

$$\mathbb{V}: (\mathbf{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Vect}.$$

We first prove

2.10.7. Lemma. *An étale map $\varphi: \Gamma \rightarrow \Gamma'$ of graphs over C induces a fully faithful functor (q.v. Definition A.1.28)*

$$G(\varphi): G(\Gamma) \rightarrow G(\Gamma')$$

between the corresponding groupoids.

Proof. We define the functor $G(\varphi)$ on objects by: $G(\varphi)(I(a)) = I(\varphi(a))$. Given an arrow $I(a) \xrightarrow{\sigma} I(b)$ in $G(\Gamma)$ we define $G(\varphi)(\sigma): I(\varphi(a)) \rightarrow I(\varphi(b))$ by

$$(2.10.8) \quad G(\varphi)\sigma := \varphi_b \circ \sigma \circ \varphi_a^{-1},$$

so that the diagram

$$(2.10.9) \quad \begin{array}{ccc} I(a) & \xrightarrow{\sigma} & I(b) \\ \downarrow \varphi_a & & \downarrow \varphi_b \\ I(\varphi(a)) & \xrightarrow{G(\varphi)\sigma} & I(\varphi(b)) \end{array}$$

commutes.⁴ Moreover, the inverse of $G(\varphi): \text{Hom}(I(a), I(b)) \rightarrow \text{Hom}(I(\varphi(a)), I(\varphi(b)))$ is given by

$$G(\varphi)^{-1}\tau = \varphi_b^{-1} \circ \tau \circ \varphi_a,$$

so $G(\varphi)$ is fully faithful. \square

2.10.10. Remark. If \bowtie is a balanced equivalence relation (see [48]) on a colored graph $\Gamma \rightarrow C$ then the quotient map $\pi: \Gamma \rightarrow \Gamma/\bowtie$ is étale. Moreover the induced map of groupoids $G(\pi)$ is not only faithful but also surjective on objects, hence an equivalence of categories by Theorem 2.10.15 below.

Proof of Theorem 2.10.6. Consider an étale map of graphs over C :

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array},$$

that is, a morphism in the category $(\text{FinGraph}/C)_{\text{ét}}$. Given an arrow $I(a) \xrightarrow{\sigma} I(b)$ in the groupoid $G(\Gamma)$ we have a commuting diagram (2.10.9) in the category of finite trees over C . Applying the control functor Ctrl gives us a commuting diagram in the category Vect of vector spaces:

$$(2.10.11) \quad \begin{array}{ccc} \text{Ctrl}(I(a)) & \xrightarrow{\text{Ctrl}(\sigma)} & \text{Ctrl}(I(b)) \\ \downarrow \text{Ctrl}(\varphi_a) & & \downarrow \text{Ctrl}(\varphi_b) \\ \text{Ctrl}(I(\varphi(a))) & \xrightarrow{\text{Ctrl}(G(\varphi)\sigma)} & \text{Ctrl}(I(\varphi(b))). \end{array}$$

Now, for any arrow $I(a) \xrightarrow{\sigma} I(b) \in G(\Gamma)$ we have the following diagram:

$$\begin{array}{ccccc} & \text{Ctrl}_{\Gamma'}(\varphi(a)) & \xrightarrow{\text{Ctrl}(\varphi_a)^{-1}} & \text{Ctrl}_{\Gamma}(a) & \\ \nearrow \varpi_{\varphi(a)} & \downarrow \text{Ctrl}(G(\varphi)\sigma) & & \downarrow \text{Ctrl}(\sigma) & \nwarrow \varpi_a \\ \mathbb{V}(\Gamma') & & & & \mathbb{V}(\Gamma) \\ \searrow \varpi_{\varphi(b)} & \downarrow \text{Ctrl}(\varphi_b)^{-1} & & \downarrow \text{Ctrl}(\varphi_b)^{-1} & \swarrow \varpi_b \\ & \text{Ctrl}_{\Gamma'}(\varphi(b)) & \xrightarrow{\text{Ctrl}(\varphi_b)^{-1}} & \text{Ctrl}_{\Gamma}(b) & \end{array}$$

⁴ Note that our definition of $G(\varphi)\sigma$ only makes sense if φ_a is an isomorphism, i.e., only if φ is étale.

where $\text{Ctrl}_\Gamma(a)$ is an abbreviation for $\text{Ctrl}(I(a))$, etc. Consequently for any arrow $I(a) \xrightarrow{\sigma} I(b) \in G(\Gamma)$ the diagram

$$\begin{array}{ccc} & \mathbb{V}(\Gamma') & \\ \varpi_{\phi(a)} \circ \text{Ctrl}(\phi_a)^{-1} \swarrow & & \searrow \varpi_{\phi(b)} \circ \text{Ctrl}(\phi_b)^{-1} \\ \text{Ctrl}_\Gamma(a) & \xrightarrow{\text{Ctrl}(\sigma)} & \text{Ctrl}_\Gamma(b) \end{array}$$

commutes. By the universal property of the limit $\mathbb{V}(\Gamma)$ there exists a unique linear map $\mathbb{V}(\varphi) : \mathbb{V}(\Gamma') \rightarrow \mathbb{V}(\Gamma)$ making the diagram

$$(2.10.12) \quad \begin{array}{ccccc} & \text{Ctrl}_{\Gamma'}(\varphi(a)) & \xrightarrow{\text{Ctrl}(\varphi_a)^{-1}} & \text{Ctrl}_\Gamma(a) & \\ \varpi_{\varphi(a)} \nearrow & & & & \nwarrow \varpi_a \\ \mathbb{V}(\Gamma') & \xrightarrow{\mathbb{V}(\varphi)} & \mathbb{V}(\Gamma) & & \\ \varpi_{\varphi(b)} \searrow & & & & \swarrow \varpi_b \\ & \text{Ctrl}_{\Gamma'}(\varphi(b)) & \xrightarrow{\text{Ctrl}(\varphi_b)^{-1}} & \text{Ctrl}_\Gamma(b) & \end{array}$$

commute for all arrows $I(a) \xrightarrow{\sigma} I(b) \in G(\Gamma)$. We leave it to the reader to check that

$$\mathbb{V}(\psi \circ \varphi) = \mathbb{V}(\varphi) \circ \mathbb{V}(\psi)$$

for any composable pair of étale maps of graphs $\Gamma \xrightarrow{\varphi} \Gamma' \xrightarrow{\psi} \Gamma''$ over C , that is, that \mathbb{V} is a contravariant functor (c.f. the proof of Proposition 6.1.3). \square

2.10.13. Example. Consider the graphs Γ , Γ' and the phase space function of Examples 2.7.4 and 2.8.5. There is an evident étale map of graphs $\varphi : \Gamma' \rightarrow \Gamma$,

$$\Gamma' = \begin{array}{c} \textcircled{1a} \\ \textcircled{1b} \end{array} \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \textcircled{2} \quad \xrightarrow{\varphi} \quad \Gamma = \textcircled{1} \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} \textcircled{2},$$

defined on the nodes by

$$\varphi(1a) = \varphi(1b) = 1, \quad \varphi(2) = 2.$$

The induced map of groupoids $G(\varphi) : G(\Gamma') \rightarrow G(\Gamma)$ sends $I(1a)$, $I(1b)$ to $I(1)$, $I(2)$ to $I(2)$, the isomorphism $I(1a) \rightarrow I(1b)$ to the identity arrow on $I(1)$ and the non-trivial arrow $\sigma : I(2) \rightarrow I(2)$ to $\sigma : I(2) \rightarrow I(2)$. In particular $G(\varphi)$ is surjective on objects. Since $G(\varphi)$ is always fully faithful (q.v. Lemma 2.10.7), it is, in this case, an equivalence of categories. Hence by Theorem 2.10.15,

$$\mathbb{V}(\varphi) : \mathbb{V}(\Gamma) \rightarrow \mathbb{V}(\Gamma')$$

is an isomorphism of vector spaces. This is not hard to check by hand: under the identifications $\mathbb{V}(\Gamma) \simeq C^\infty(V \times V, V)^{\mathbb{Z}_2}$ and $\mathbb{V}(\Gamma') \simeq C^\infty(V \times V, V)^{\mathbb{Z}_2}$, the map $\mathbb{V}(\varphi)$ is the identity map:

$$\mathbb{V}(\varphi)f = f$$

for all $f \in C^\infty(V \times V, V)^{\mathbb{Z}_2}$.

2.10.14. Remark. Our construction of the functor \mathbb{V} (Definition 2.7.2 and Theorem 2.10.6) that assigns to each finite graph a space of virtual groupoid-invariant vector fields is analogous to our construction of the phase space functor \mathbb{P} . In both cases we have a fixed category \mathbf{D} with finite

limits and a collection of categories \mathcal{U} ; we then form a new category \mathbb{U} with objects of the form $U \xrightarrow{f} D$, where $U \in \mathcal{U}$ and f a functor, and show that the assignment

$$\mathbb{U}_0 \rightarrow \mathbf{D}_0, \quad (U \xrightarrow{f} D) \mapsto \lim(U \xrightarrow{f} D)$$

extends to a functor.

In the case of \mathbb{P} , we took $\mathbf{D} = \mathbf{Euc}$, the category of Euclidean spaces and smooth maps, and \mathcal{U} as the collection of all finite sets over a fixed set C_0 . In the case of \mathbb{V} , we took $\mathbf{D} = \mathbf{Vect}$, the category of vector spaces and linear maps, and \mathcal{U} as the collection of all groupoids of the form $G(\Gamma)$ with Γ a finite graph over the fixed graph of colors C .

The essential difference between the two constructions is that while in the case of the phase space functor \mathbb{P} the morphisms of \mathbb{U} are, roughly, commuting triangles

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \searrow & & \swarrow \beta \\ & \mathbf{D} & \end{array},$$

in the case of the construction of the functor \mathbb{V} the morphisms are more complicated: the triangles of the form

$$\begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ \text{Ctrl}|_{G(\Gamma)} \searrow & & \swarrow \text{Ctrl}|_{G(\Gamma')} \\ & \mathbf{Vect} & \end{array}$$

no longer strictly commute. Instead they “2-commute”—there is a natural isomorphism

$$\text{Ctrl}|_{G(\Gamma)} \xrightarrow{\phi} \text{Ctrl}|_{G(\Gamma')} \circ G(\varphi)$$

—this is the content of (2.10.11): the components of ϕ are linear isomorphisms $\text{Ctrl}(\varphi_a)^{-1}$. A standard notation for such a 2-commutative diagram is:

$$\begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ \text{Ctrl}|_{G(\Gamma)} \searrow & \xRightarrow{\phi} & \swarrow \text{Ctrl}|_{G(\Gamma')} \\ & \mathbf{Vect} & \end{array}.$$

A category-theoretic framework that encompasses both constructions is discussed in section 6. Two interesting results that follow from this framework are:

2.10.15. Theorem. *Suppose the map $G(\varphi): G(\Gamma) \rightarrow G(\Gamma')$ induced by an étale map $\varphi: \Gamma \rightarrow \Gamma'$ of graphs over C is essentially surjective. Then the map*

$$\mathbb{V}(\varphi): \mathbb{V}(\Gamma') \rightarrow \mathbb{V}(\Gamma)$$

is an isomorphism.

In particular, if \bowtie is a balanced equivalence relation [49] on Γ and $\pi: \Gamma \rightarrow \Gamma/\bowtie$ is the quotient map (q.v. Remark 2.10.10), then

$$\mathbb{V}(\pi): \mathbb{V}(\Gamma/\bowtie) \rightarrow \mathbb{V}(\Gamma)$$

*is an isomorphism.*⁵

⁵In the language of [48] the graphs Γ and Γ/\bowtie are “ODE equivalent.”

2.10.16. For an input tree $I(a)$ of a graph Γ over C let

$$\text{Aut}(I(a)) = \text{Hom}_{\text{Graph}/C}(I(a), I(a)),$$

the group of isomorphisms of a colored tree $I(a)$. Then the skeleton of a groupoid $G(\Gamma)$ (q.v. A.1.35) is a disjoint union of groups of the form

$$\text{Aut}(I(a_1)) \sqcup \dots \sqcup \text{Aut}(I(a_k))$$

for some nodes a_1, \dots, a_k of Γ .

2.10.17. **Theorem.** *For any colored graph $\Gamma \rightarrow C$ the space of virtual invariant vector fields $\mathbb{V}(\Gamma)$ is a product of invariants of groups. In particular let*

$$\text{Aut}(I(a_1)) \sqcup \dots \sqcup \text{Aut}(I(a_k))$$

be a skeleton of the groupoid $G(\Gamma)$. Then

$$\mathbb{V}(\Gamma) \simeq \prod_{i=1}^k \text{Ctrl}(I(a_i))^{\text{Aut}(I(a_i))}.$$

The proofs of 2.10.15 and 2.10.17 are given in Section 6.

2.10.18. **Remark.** Note that while the spaces $\mathbb{V}(\Gamma), \mathbb{V}(\Gamma')$ of virtual groupoid-invariant vector fields in the Example 2.10.13 above are naturally isomorphic, there is no evident natural isomorphism between their images $S_\Gamma(\mathbb{V}(\Gamma))$ and $S_{\Gamma'}(\mathbb{V}(\Gamma'))$ in the spaces of vector fields $\chi(\mathbb{P}\Gamma_0)$ and $\chi(\mathbb{P}\Gamma'_0)$ respectively. Indeed, as we computed in Example 2.8.5,

$$S_\Gamma(\mathbb{V}(\Gamma)) = \{(0, g) \in C^\infty(V^2, V^2) \mid g(v_1, v_2) = f(v_1, v_1) \text{ for some } f \in C^\infty(V \times V, V)^{\mathbb{Z}_2}\}$$

while

$$S_{\Gamma'}(\mathbb{V}(\Gamma')) = \{(0, 0, g) \in C^\infty(V^3, V^3) \mid g(v_{1a}, v_{1b}, v_2) = f(v_{1a}, v_{1b}) \text{ for some } f \in C^\infty(V \times V, V)^{\mathbb{Z}_2}\}.$$

As we have seen in Example 2.8.5, $S_{\Gamma'}$ is injective. So its image is isomorphic to $C^\infty(V \times V, V)^{\mathbb{Z}_2}$:

$$S_{\Gamma'}(\mathbb{V}(\Gamma')) \simeq C^\infty(V \times V, V)^{\mathbb{Z}_2}.$$

On the other hand the map

$$C^\infty(V \times V, V)^{\mathbb{Z}_2} \rightarrow C^\infty(V, V), f \mapsto (g(v) = f(v, v))$$

is surjective. Indeed for any $g \in C^\infty(V, V)$

$$g(v) = f(v, v)$$

where $f(u, w) = \frac{1}{2}(g(u) + g(w))$. Consequently

$$S_\Gamma(\mathbb{V}(\Gamma)) \simeq C^\infty(V, V),$$

and there is no natural isomorphism between $C^\infty(V, V)$ and $C^\infty(V \times V, V)^{\mathbb{Z}_2}$. For this reason we are confused by the definition of the pullback map β^* on p. 332 of [49]. Consequently we do not fully understand Definition 6.1 (op. cit.) of G -admissible vector fields.

While there are no natural geometric or combinatorial maps between the *images* $S_\Gamma(\mathbb{V}(\Gamma))$ and $S_{\Gamma'}(\mathbb{V}(\Gamma'))$ for any étale map of graphs $\varphi : \Gamma' \rightarrow \Gamma$ there is a very interesting relationship between the maps $\mathbb{V}(\varphi), \mathbb{P}\varphi, S_\Gamma$ and $S_{\Gamma'}$. This relationship is the subject of the next two subsections.

2.11. Virtual groupoid invariant vector fields and dynamics on Euclidean spaces.

Let $\varphi: \Gamma \rightarrow \Gamma'$ be an étale map of finite graphs over our graph of colors C . Let $w \in \mathbb{V}(\Gamma')$ be a groupoid-invariant virtual vector field on the phase space $\mathbb{P}\Gamma'_0$. Then $S_{\Gamma'}w$ is an actual vector field on $\mathbb{P}\Gamma'_0$. Since the phase space functor is contravariant, we have a smooth map of phase spaces $\mathbb{P}\varphi: \mathbb{P}\Gamma'_0 \rightarrow \mathbb{P}\Gamma_0$. Since the functor \mathbb{V} is contravariant, we have a virtual vector field $\mathbb{V}(\varphi)w \in \mathbb{V}(\Gamma)$, which gives us an actual vector field $S_{\Gamma}(\mathbb{V}(\varphi)w)$ on $\mathbb{P}\Gamma_0$. Remarkably the vector fields $S_{\Gamma'}w$ and $S_{\Gamma}(\mathbb{V}(\varphi)w)$ are $\mathbb{P}\varphi$ -related, which is the content of the Theorem 2.11.2 below.

2.11.1. Remark. Note that if Γ' is the quotient of the graph Γ by a balanced equivalence relation and φ is the quotient map, then $\mathbb{P}\varphi$ identifies $\mathbb{P}\Gamma'$ with a “polydiagonal” subspace of $\mathbb{P}\Gamma$. The fact that the vector fields $S_{\Gamma'}w$ and $S_{\Gamma}(\mathbb{V}(\varphi)w)$ are $\mathbb{P}\varphi$ -related translates into: “the restriction of $S_{\Gamma}(\mathbb{V}(\varphi)w)$ to the polydiagonal is $S_{\Gamma'}w$.” Thus Theorem 2.11.2 generalizes the synchrony results of [46] and [48] from quotient maps of colored graphs to arbitrary étale maps of colored graphs.

2.11.2. Theorem. *Let $w \in \mathbb{V}(\Gamma')$ be a virtual dynamical system on $\mathbb{P}\Gamma_0$. For each étale map of graphs*

$$\varphi: \Gamma \rightarrow \Gamma',$$

the diagram

$$(2.11.3) \quad \begin{array}{ccc} \mathbb{P}(\Gamma'_0) & \xrightarrow{D\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \\ S_{\Gamma'}(w) \uparrow & & \uparrow S_{\Gamma}(\mathbb{V}(\varphi)w) \\ \mathbb{P}(\Gamma'_0) & \xrightarrow{\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \end{array}$$

commutes: for any point $x \in \mathbb{P}(\Gamma'_0)$

$$(2.11.4) \quad D\mathbb{P}\varphi(x) (S_{\Gamma'}(w)(x)) = S_{\Gamma}(\mathbb{V}(\varphi)w)(\mathbb{P}\varphi(x)).$$

In other words,

$$\mathbb{P}(\varphi): (\mathbb{P}(\Gamma'), S_{\Gamma'}(w)) \rightarrow (\mathbb{P}(\Gamma), S_{\Gamma}(\mathbb{V}(\varphi)w))$$

is a map of dynamical systems.

2.11.5. Remark. Since the map $\mathbb{P}\varphi$ above is linear (q.v. 2.3.11), $D\mathbb{P}\varphi(x) = \mathbb{P}\varphi$ for all $x \in \mathbb{P}(\Gamma'_0)$. Therefore (2.11.4) amounts to:

$$(2.11.6) \quad \mathbb{P}\varphi(S_{\Gamma'}(w)(x)) = S_{\Gamma}(\mathbb{V}(\varphi)w)(\mathbb{P}\varphi(x)) \quad \text{for any point } x \in \mathbb{P}(\Gamma'_0).$$

Proof of Theorem 2.11.2. Recall that for each node a of a graph Γ over the graph of colors C we have a canonical map $\xi: I(a) \rightarrow \Gamma$ from the input tree $I(a)$ of the node $a \in \Gamma_0$ to the graph Γ (q.v. 2.6.2; we use the same letter ξ for all nodes of Γ and for all graphs over C suppressing both dependencies and trusting that lighter notation will cause no confusion).

If $\varphi: \Gamma \rightarrow \Gamma'$ is an étale map of graphs over C , then by the definition of “étale”, for each node a of Γ , we have an induced *isomorphism* of graphs over C :

$$\varphi_a: I(a) \rightarrow I(\varphi(a))$$

defined by (2.10.1). Note that φ_a necessarily sends the root a of $I(a)$ to the root $\varphi(a)$ of $I(\varphi(a))$ and the leaves $\text{lv } I(a)$ bijectively to the leaves $\text{lv } I(\varphi(a))$ of $I(\varphi(a))$. Moreover, the diagram of

graphs over C

$$(2.11.7) \quad \begin{array}{ccc} I(a) & \xrightarrow{\xi} & \Gamma \\ \downarrow \varphi_a & & \downarrow \varphi \\ I(\varphi(a)) & \xrightarrow{\xi} & \Gamma' \end{array}$$

commutes. Hence we have a commuting diagram

$$(2.11.8) \quad \begin{array}{ccc} \text{lv } I(a) & \xrightarrow{\xi} & \Gamma_0 \\ \downarrow \varphi_a & & \downarrow \varphi \\ \text{lv } I(\varphi(a)) & \xrightarrow{\xi} & \Gamma'_0 \end{array}$$

of finite sets over C_0 and consequently, since \mathbb{P} is a functor, the commuting diagram

$$(2.11.9) \quad \begin{array}{ccc} \mathbb{P} \text{lv } I(a) & \xleftarrow{\mathbb{P}\xi} & \mathbb{P}\Gamma_0 \\ \uparrow \mathbb{P}\varphi_a & & \uparrow \mathbb{P}\varphi \\ \mathbb{P} \text{lv } I(\varphi(a)) & \xleftarrow{\mathbb{P}\xi} & \mathbb{P}\Gamma'_0 \end{array}$$

in our category \mathbf{Euc} of Euclidean phase spaces. Let

$$(2.11.10) \quad \kappa_a: \{a\} \hookrightarrow \Gamma_0$$

denote the canonical inclusion in the category \mathbf{FinSet}/C_0 of finite sets over C_0 . We then have a commuting diagram in \mathbf{FinSet}/C_0

$$(2.11.11) \quad \begin{array}{ccc} \{a\} & \xrightarrow{\kappa_a} & \Gamma_0 \\ \downarrow \varphi_a|_{\{a\}} = \varphi|_{\{a\}} & & \downarrow \varphi \\ \{\varphi(a)\} & \xrightarrow{\kappa_{\varphi(a)}} & \Gamma'_0, \end{array}$$

hence a commuting diagram in \mathbf{Euc} :

$$(2.11.12) \quad \begin{array}{ccc} \mathbb{P}\{a\} & \xleftarrow{\mathbb{P}\kappa_a} & \mathbb{P}\Gamma_0 \\ \uparrow \mathbb{P}(\varphi|_{\{a\}}) & & \uparrow \mathbb{P}\varphi \\ \mathbb{P}\{\varphi(a)\} & \xleftarrow{\mathbb{P}\kappa_{\varphi(a)}} & \mathbb{P}\Gamma'_0, \end{array}$$

Differentiating (2.11.12) we get the commuting diagram

$$(2.11.13) \quad \begin{array}{ccc} T\mathbb{P}\{a\} & \xleftarrow{D\mathbb{P}\kappa_a} & T\mathbb{P}\Gamma_0 \\ \uparrow D\mathbb{P}(\varphi|_{\{a\}}) & & \uparrow D\mathbb{P}\varphi \\ T\mathbb{P}\{\varphi(a)\} & \xleftarrow{D\mathbb{P}\kappa_{\varphi(a)}} & T\mathbb{P}\Gamma'_0, \end{array}$$

of tangent bundles. Recall that the projection

$$p_a: \chi(\mathbb{P}\Gamma_0) \rightarrow \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\})$$

from (2.8.4) is given by

$$(2.11.14) \quad p_a(X) = D(\mathbb{P}\kappa_a) \circ X.$$

By definition of the maps S_Γ and $S_{\Gamma'}$ (q.v. (2.8.2)) we have

$$(2.11.15) \quad D\mathbb{P}\kappa_a \circ S_\Gamma(v) = \varpi_a(v) \circ \mathbb{P}(\xi|_{\text{lv } I(a)}) \quad \text{for any } a \in \Gamma_0 \text{ and any } v \in \mathbb{V}(\Gamma)$$

and

$$(2.11.16) \quad D\mathbb{P}\kappa_{\varphi(a)} \circ S_{\Gamma'}(v') = \varpi_{\varphi(a)}(v') \circ \mathbb{P}(\xi|_{\text{lv } I(\varphi(a))}) \quad \text{for any } a \in \Gamma_0 \text{ and any } v' \in \mathbb{V}(\Gamma').$$

By definition (2.10.12) of $\mathbb{V}\varphi$

$$(2.11.17) \quad \varpi_a(\mathbb{V}(\varphi)w) = \text{Ctrl}(\varphi_a)^{-1}(\varpi_{\varphi(a)}w)$$

for any virtual invariant vector field $w \in \mathbb{V}(\Gamma')$ and any node $a \in \Gamma_0$. By definition of $\text{Ctrl}(\varphi_a)$ (q.v. (2.5.3), (2.5.4))

$$\text{Ctrl}(\varphi_a)^{-1}u = u \circ (\mathbb{P}\varphi_a)^{-1} \quad \text{for any } u \in \text{Ctrl}(I(\varphi(a))).$$

Hence

$$\varpi_a(\mathbb{V}(\varphi)w) = (\varpi_{\varphi(a)}w) \circ (\mathbb{P}\varphi_a)^{-1}.$$

Consequently

$$(2.11.18) \quad \varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi_a = \varpi_{\varphi(a)}w.$$

Since the vector space of vector fields $\chi(\mathbb{P}\Gamma_0)$ is the product

$$\chi(\mathbb{P}\Gamma_0) = \prod_{a \in \Gamma_0} \text{Hom}_{\text{Euc}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}),$$

the diagram (2.11.3) commutes if and only if

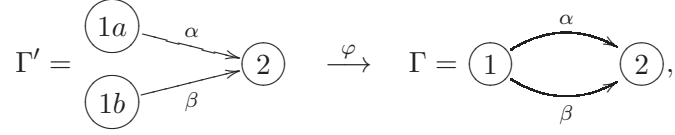
$$p_a(S_\Gamma(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi) = p_a(D\mathbb{P}\varphi \circ S_{\Gamma'}(w))$$

for every $a \in \Gamma_0$. We now compute:

$$\begin{aligned} p_a(S_\Gamma(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi) &= D\mathbb{P}\kappa_a \circ S_\Gamma(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi, \quad \text{by (2.11.14)} \\ &= (\varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}(\xi|_{\text{lv } I(a)})) \circ \mathbb{P}\varphi, \quad \text{by (2.11.15)} \\ &= \varpi_a(\mathbb{V}(\varphi)w) \circ (\mathbb{P}(\xi|_{\text{lv } I(a)}) \circ \mathbb{P}\varphi) \\ &= \varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi_a \circ \mathbb{P}(\xi|_{\text{lv } I(\varphi(a))}) \quad \text{by (2.11.9)} \\ &= \varpi_{\varphi(a)}w \circ \mathbb{P}(\xi|_{\text{lv } I(\varphi(a))}), \quad \text{by (2.11.18)} \\ &= D\mathbb{P}\kappa_{\varphi(a)} \circ S_{\Gamma'}(w) \quad \text{by (2.11.16)} \\ &= D\mathbb{P}(\varphi|_{\{a\}}) \circ D\mathbb{P}\kappa_{\varphi(a)} \circ S_{\Gamma'}(w) \quad \text{since } D\mathbb{P}(\varphi|_{\{a\}}) = D(id_{\mathbb{P}\{a\}}) = id \\ &= D\mathbb{P}\kappa_a \circ D\mathbb{P}\varphi \circ S_{\Gamma'}(w), \quad \text{since (2.11.13) commutes} \\ &= p_a(D\mathbb{P}\varphi(S_{\Gamma'}(w))). \end{aligned}$$

And we are done. \square

2.11.19. Example. Here we revisit Examples 2.7.4, 2.8.5 and 2.10.13. Recall that we have an étale map of (mono-colored) graphs $\varphi : \Gamma' \rightarrow \Gamma$:



and the phase space function $\mathcal{P}(\circ) = V$. Then $\mathbb{P}\Gamma_0 = V \times V =: V^2$, $\mathbb{P}(\Gamma'_0) = V \times V \times V =: V^3$ and $\mathbb{P}\varphi : V^2 \rightarrow V^3$ is given by

$$(\mathbb{P}\varphi)(v_1, v_2) = (v_1, v_1, v_2)$$

(cf. Examples 2.3.6 and 2.3.7). Recall that $\mathbb{V}(\Gamma) \simeq \mathbb{C}^\infty(V \times V, V)^{\mathbb{Z}_2} \simeq \mathbb{V}(\Gamma')$ and that under these identifications

$$\mathbb{V}(\varphi) : \mathbb{V}(\Gamma) \rightarrow \mathbb{V}(\Gamma')$$

is the identity map. Recall also that for any $f \in \mathbb{C}^\infty(V^2, V)^{\mathbb{Z}_2}$

$$S_\Gamma(f)(v_1, v_2) = (0, f(v_1, v_1))$$

while

$$S_{\Gamma'}(f)(v_{1a}, v_{1b}, v_2) = (0, f(v_{1a}, v_{1b})).$$

Consequently

$$((\mathbb{P}\varphi) \circ (S_\Gamma f))(v_1, v_2) = (0, 0, f(v_1, v_1)) = ((S_{\Gamma'} f) \circ (\mathbb{P}\varphi))(v_1, v_2),$$

which is precisely the assertion of Theorem 2.11.2. Thus the “polydiagonal” $\mathbb{P}\varphi(V^2) \subset V^3$ is an invariant submanifold for any “groupoid-invariant” vector field $v \in S_{\Gamma'}(\mathbb{V}(\Gamma')) \subset \chi(V^3)$.

2.12. The combinatorial category \mathcal{V}^{Euc} of dynamical systems.

Having constructed a contravariant functor $\mathbb{V} : (\text{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Vect}$ that assigns to each C -colored graph Γ the vector space of virtual groupoid-invariant vector fields on the phase space $\mathbb{P}\Gamma_0$, we now construct a category \mathcal{V}^{Euc} whose set of objects is the collection of all groupoid-invariant vector fields $\bigsqcup_{\Gamma \in \text{FinGraph}/C} \mathbb{V}(\Gamma)$. This construction is, incidentally, an example of the construction of the category of elements of a presheaf.

2.12.1. Definition (The combinatorial category \mathcal{V}^{Euc} of dynamical systems). Fix a graph of colors $C = \{C_1 \rightrightarrows C_0\}$ and a phase space function $\mathcal{P} : C_0 \rightarrow \mathbf{Euc}$. Let $\mathbb{P} : \text{FinSet}/C_0 \rightarrow \mathbf{Euc}$ be the associated phase space functor (2.3.3) and $\mathbb{V} : (\text{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Vect}$ the corresponding virtual vector fields functor.

- (1) The collection of objects $(\mathcal{V}^{\text{Euc}})_0$ is the collection of pairs

$$(\mathcal{V}^{\text{Euc}})_0 := \{(v, \Gamma) \mid \Gamma \in (\text{FinGraph}/C)_0, v \in \mathbb{V}(\Gamma)\} = \bigsqcup_{\Gamma \in (\text{FinGraph}/C)_0} \mathbb{V}(\Gamma).$$

- (2) For any two objects $(v, \Gamma), (v', \Gamma') \in \mathcal{V}^{\text{Euc}}_0$, the set of morphisms $\text{Hom}_{\mathcal{V}^{\text{Euc}}}((v, \Gamma), (v', \Gamma'))$ is given by

$$\text{Hom}_{\mathcal{V}^{\text{Euc}}}((v, \Gamma), (v', \Gamma')) := \left\{ (h, (v, \Gamma), (v', \Gamma')) \in \text{Hom}_{(\text{FinGraph}/C)_{\text{et}}}(\Gamma, \Gamma') \times (\mathcal{V}^{\text{Euc}})_0 \times (\mathcal{V}^{\text{Euc}})_0 \mid \mathbb{V}(h)v = v' \right\}.$$

Composition in \mathcal{V}^{Euc} is defined by

$$(g, (v', \Gamma'), (v'', \Gamma'')) \circ (h, (v, \Gamma), (v', \Gamma')) = (hg, (v, \Gamma), (v'', \Gamma''))$$

for all $\Gamma \xleftarrow{h} \Gamma' \xleftarrow{g} \Gamma''$ and $v \in \mathbb{V}\Gamma, v' \in \mathbb{V}\Gamma', v'' \in \mathbb{V}\Gamma''$ with $\mathbb{V}(g)v' = v''$ and $\mathbb{V}(h)v = v'$.

2.12.2. Remark. There is an evident functor $\pi: \mathcal{V}^{\text{Euc}} \rightarrow (\text{FinGraph}/C)_{\text{et}}^{\text{op}}$. On objects $\pi(v, \Gamma) = \Gamma$ and on morphisms

$$\pi \left((v, \Gamma) \xrightarrow{(h, (v, \Gamma), (v', \Gamma'))} (v', \Gamma') \right) = \Gamma \xleftarrow{h} \Gamma' \in \text{Hom}_{(\text{FinGraph}/C)_{\text{et}}^{\text{op}}}(\Gamma, \Gamma').$$

As we mention in the introduction, there is a category **Dynamical Systems** whose objects are pairs (M, X) where M is a manifold and X a vector field on M . A morphism $\varphi: (M, X) \rightarrow (M', X')$ in **Dynamical Systems** is a smooth map of manifolds $\varphi: M \rightarrow M'$ with

$$d\varphi \circ X = X' \circ \varphi,$$

that is, a map of control systems (q.v. 2.1.8). There is an evident functor

$$U: \text{Dynamical Systems} \rightarrow \text{Man}$$

from the category of dynamical systems to the category of manifolds that forgets the vector fields. The category **Dynamical Systems** has a full subcategory **Dynamical Systems**_{Euc} whose objects are pairs (W, X) where W is a Euclidean space. By construction

$$U(\text{Dynamical Systems}_{\text{Euc}}) = \text{Euc}.$$

Theorem 2.11.2 allows us to reinterpret the collection of maps $\{S_\Gamma: \mathbb{V}\Gamma \rightarrow \chi(\mathbb{P}\Gamma_0)\}_{\Gamma \in (\text{FinGraph}/C)_0}$ as a functor

$$S: \mathcal{V}^{\text{Euc}} \rightarrow \text{Dynamical Systems}_{\text{Euc}}.$$

Indeed, given an object $(v, \Gamma) \in \mathcal{V}^{\text{Euc}}$ we define

$$S(v, \Gamma) = (\mathbb{P}\Gamma_0, S_\Gamma(v)),$$

where, as before, \mathbb{P} denotes the phase space functor and $S_\Gamma: \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0)$ is defined by (2.8.2) (see also 2.8.1). Given a morphism $(h, (v, \Gamma), (v', \Gamma')) \in \mathcal{V}_{\text{Euc}}$ we define

$$S(h, (v, \Gamma), (v', \Gamma')) = (\mathbb{P}\Gamma_0, S_\Gamma(v)) \xrightarrow{\mathbb{P}h} (\mathbb{P}\Gamma', S_{\Gamma'}(v')).$$

By Theorem 2.11.2

$$D\mathbb{P}h(S_\Gamma(v)) = S_{\Gamma'}(v') \circ \mathbb{P}h.$$

Therefore

$$\mathbb{P}h: (\mathbb{P}\Gamma_0, S_\Gamma(v)) \rightarrow (\mathbb{P}\Gamma', S_{\Gamma'}(v'))$$

is a morphism of dynamical systems. We have thus proved:

2.12.3. Theorem. Let \mathcal{V}^{Euc} denote the category of virtual groupoid-invariant dynamical systems constructed above, $\pi: \mathcal{V}^{\text{Euc}} \rightarrow (\text{FinGraph}/C)_{\text{et}}^{\text{op}}$ the canonical projection, $U: \text{Dynamical Systems}_{\text{Euc}} \rightarrow \text{Euc}$ the forgetful functor, and \mathbb{P} the phase space functor extended to finite graphs (q.v. 2.3.14). There exists a functor $S: \mathcal{V}^{\text{Euc}} \rightarrow \text{Dynamical Systems}_{\text{Euc}}$ making the diagram

$$(2.12.4) \quad \begin{array}{ccc} \mathcal{V}^{\text{Euc}} & \xrightarrow{S} & \text{Dynamical Systems}_{\text{Euc}} \\ \pi \downarrow & & \downarrow U \\ (\text{FinGraph}/C)_{\text{et}}^{\text{op}} & \xrightarrow{\mathbb{P}} & \text{Euc} \end{array}$$

commute. On objects

$$S(v, \Gamma) = (\mathbb{P}\Gamma_0, S_\Gamma(v));$$

on arrows

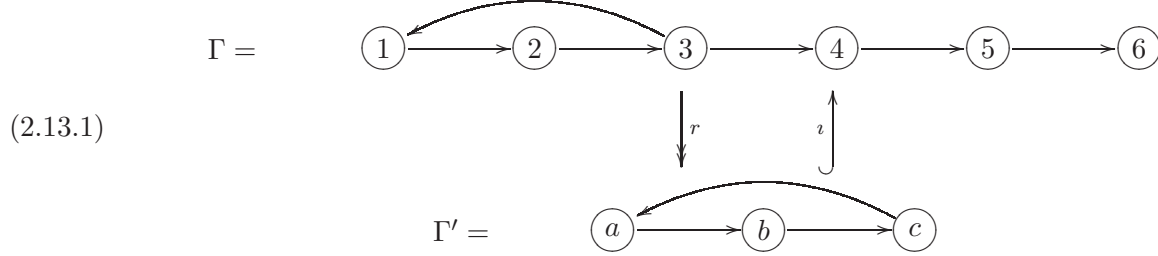
$$S(h, (v, \Gamma), (v', \Gamma')) = (\mathbb{P}\Gamma_0, S_\Gamma(v)) \xrightarrow{\mathbb{P}h} (\mathbb{P}\Gamma', S_{\Gamma'}(v)).$$

2.13. An example.

Let us consider the two graphs Γ, Γ' over the graph of colors $C = \circ \curvearrowright \circ$ depicted in (2.13.1) below, along with the maps $r: \Gamma \rightarrow \Gamma', \iota: \Gamma' \rightarrow \Gamma$ given on nodes by

$$r(1, 2, 3, 4, 5, 6) = (a, b, c, a, b, c), \quad \iota(a, b, c) = (1, 2, 3).$$

Clearly both r and ι are étale, r is surjective and ι is injective, and $r \circ \iota = id_{\Gamma'}$.



Let Euclidean space V be the value of the space function \mathcal{P} on the only node of the graph C of colors:

$$V = \mathcal{P}(\circ).$$

Then the phase space associate to any node of Γ and Γ' is V .

We first consider the graph Γ . Define the function $\ell: [6] \rightarrow [6]$ by $\ell(1, 2, 3, 4, 5, 6) = (3, 1, 2, 3, 4, 5)$, i.e. $\ell(i)$ is the number of the node which precedes the node i in the graph Γ . We label the unique arrow of Γ whose target is the node i by $\gamma_{\ell(i), i}$. Consequently each input tree of Γ is of the form

$$I(i) = \begin{array}{c} \circlearrowleft \\ i \end{array} \xleftarrow{\gamma_{\ell(i), i}} \{ \gamma_{\ell(i), i} \}.$$

Thus $\text{Ctrl}(I(i)) = C^\infty(V, V)$. We next consider the groupoid $G(\Gamma)$. For any nodes i, j of Γ , the map $\sigma_{ij}: I(i) \rightarrow I(j)$, defined by $\sigma_{ij}: (i \mapsto j, \gamma_{\ell(i), i} \mapsto \gamma_{\ell(j), j})$ is an isomorphism of input trees. Since $\sigma_{ij}|_{\text{lv } I(i)}$ is the unique map sending the one point set $\{\ell(i)\}$ to the one point set $\{\ell(j)\}$, $\mathbb{P}(\sigma_{ij}|_{\text{lv } I(i)}): \mathbb{P} \text{lv } I(j) \rightarrow \mathbb{P} \text{lv } I(i)$ is the identity map for all nodes i, j . Consequently

$$\text{Ctrl}(\sigma_{ij}): \text{Ctrl}(I(i)) \rightarrow \text{Ctrl}(I(j)),$$

which is defined by

$$\text{Ctrl}(\sigma_{ij})(f) = id \circ f \circ \mathbb{P}\sigma_{ij}$$

for any $f \in \text{Ctrl}(I(i)) = C^\infty(\mathbb{P} \text{lv } I(i), \mathbb{P}\{i\}) = C^\infty(V, V)$, is the identity map. Informally speaking the j th component of the groupoid invariant vector field f on $\mathbb{P}\Gamma_0$ depends on $\mathbb{P}\{\ell(i)\} = \mathbb{P} \text{lv } I(i)$ in the same way that the i th component of f depends on $\mathbb{P}\{\ell(i)\} = \mathbb{P} \text{lv } I(i)$. More formally by Theorem 2.10.17

$$\mathbb{V}(\Gamma) \simeq C^\infty(V, V).$$

Moreover it is not hard to see that $S_\Gamma: \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0) = \chi(V^6)$ is given by

$$S_\Gamma(f)(v_1, \dots, v_6) = (f(v_3), f(v_1), f(v_2), f(v_3), f(v_4), f(v_5))$$

for all $f \in C^\infty(V, V)$. Similarly, $\mathbb{P}(\Gamma'_0) = V^3$,

$$\mathbb{V}(\Gamma') \simeq C^\infty(V, V).$$

and $S_{\Gamma'}: \mathbb{V}(\Gamma') \rightarrow \chi(\mathbb{P}\Gamma'_0) = \chi(V^3)$ is given by

$$(2.13.2) \quad S_{\Gamma'}(g)(v_a, v_b, v_c) = (g(x_c), g(x_a), g(x_b))$$

Using (2.3.8) we see that

$$\mathbb{P}r: V^3 = \mathbb{P}\Gamma'_0 \rightarrow \mathbb{P}\Gamma = \mathbb{V}^6$$

is an embedding given by

$$\mathbb{P}r(v_a, v_b, v_c) = (v_a, v_b, v_c, v_a, v_b, v_c),$$

and $\mathbb{P}i: V^6 \rightarrow V^3$ is the projection given by

$$\mathbb{P}i(v_1, v_2, v_3, v_4, v_5, v_6) = (v_1, v_2, v_3)$$

Note also that

$$\mathbb{P}(i \circ r) = \mathbb{P}r \circ \mathbb{P}i: V^6 \rightarrow V^6$$

is the submersion onto the diagonal $\Delta_{V^3} \subset V^3 \times V^3 \simeq V^6$ given by

$$\mathbb{P}(i \circ r)(v_1, v_2, v_3, v_4, v_5, v_6) = (v_1, v_2, v_3, v_1, v_2, v_3).$$

We now turn to the maps $\mathbb{V}(r)$ and $\mathbb{V}(i)$ between the spaces of virtual groupoid invariant vector fields. For any node i of Γ the map

$$r_i: I(i) \rightarrow I(r(i))$$

is an isomorphism inducing the identity map

$$\text{Ctrl}(r_i): C^\infty(V, V) = \text{Ctrl}(I(i)) \rightarrow \text{Ctrl}(I(r(i))) = C^\infty(V, V).$$

It follows from the construction of the functor \mathbb{V} on arrows (q.v. (2.10.12), (2.11.17)) that

$$\mathbb{V}(r): \mathbb{V}(\Gamma') \rightarrow \mathbb{V}(\Gamma)$$

is the identity map on $C^\infty(V, V)$. Similarly $\mathbb{V}(i)$ is the identity map as well.

By Theorem 2.11.2 for any virtual groupoid invariant vector field $f \in \mathbb{V}(\Gamma)$ we have

$$(2.13.3) \quad D\mathbb{P}(i \circ r) S_\Gamma(f) = S_\Gamma(\mathbb{V}(i \circ r)f) \circ \mathbb{P}(i \circ r)$$

Recall that for any map φ of graphs over C the map of Euclidean spaces $\mathbb{P}(\varphi)$ is linear and consequently $D\mathbb{P}\varphi = \mathbb{P}\varphi$ (q.v. 2.11.5). Therefore (2.13.3) amounts to

$$\mathbb{P}(i \circ r) S_\Gamma(f) = S_\Gamma(\mathbb{V}(i \circ r)f) \circ \mathbb{P}(i \circ r),$$

c.f. (2.11.6), or, in this case,

$$\mathbb{P}(i \circ r) \circ S_\Gamma(f) = S_\Gamma(f) \circ \mathbb{P}(i \circ r).$$

Therefore the diagonal $\Delta_{V^3} \subset \mathbb{P}\Gamma_0$ is an invariant submanifold for any groupoid invariant vector field $F \in S_\Gamma(\mathbb{V}(\Gamma))$. Moreover, $\mathbb{P}(i \circ r)$ projects the dynamics of F on $\mathbb{P}\Gamma_0$ onto the dynamics of $F|_{\Delta_{V^3}}$. Thus for an equilibrium $x \in \Delta_{V^3}$ of $F|_{\Delta_{V^3}}$, the set $(\mathbb{P}(i \circ r))^{-1}(x)$ is an invariant submanifold of F . To see this, if we have $S_\Gamma(f)x = 0$ and $\mathbb{P}(i \circ r)y = x$, then

$$\mathbb{P}(i \circ r) S_\Gamma(f)y = S_\Gamma(f)\mathbb{P}(i \circ r)y = S_\Gamma(f)x = 0,$$

so that $S_\Gamma(f)$ must be tangent to the diagonal Δ_{V^3} . Similarly, for a periodic orbit O of $F|_{\Delta_{V^3}}$, the set $(\mathbb{P}(i \circ r))^{-1}(O)$ is a “relative periodic orbit” of F , and so on.

By Theorem 2.11.2 again, a dynamical system of the form $(\Delta_{V^3}, F|_{\Delta_{V^3}}, F \in S_\Gamma(\mathbb{V}(\Gamma)))$ is an isomorphic image (under $\mathbb{P}(r)$) of a dynamical system of the form $(\mathbb{P}(\Gamma'_0), G, G \in S_{\Gamma'}(\mathbb{V}(\Gamma')))$. By Theorem 5.1.5 and Example 5.1.6 any groupoid invariant vector field $G \in S_{\Gamma'}(\mathbb{V}(\Gamma'))$ is also invariant under the action of the cyclic group \mathbb{Z}_3 . This can also be seen directly from (2.13.2). Thus we can use the machinery of equivariant dynamical systems to study the dynamics of groupoid invariant vector fields on Δ_{V^3} and thereby on $\mathbb{P}\Gamma_0$.

Finally note that the “color” maps $\mathbf{c}: \Gamma \rightarrow C = \bigcirc$ and $\mathbf{c}': \Gamma' \rightarrow C$ are both étale. Then by Theorem 2.11.2 the maps $\mathbb{P}\mathbf{c}: V = \mathbb{P}C_0 \rightarrow \mathbb{P}\Gamma_0$ and $\mathbb{P}\mathbf{c}': V \rightarrow \mathbb{P}\Gamma'_0$ define invariant subsystems for any groupoid invariant vector fields $F \in S_\Gamma(\mathbb{V}(\Gamma))$ and $G \in S_{\Gamma'}(\mathbb{V}(\Gamma'))$, respectively.

3. LINEAR GROUPOID INVARIANT VECTOR FIELDS

3.1.

The functor $\mathbb{V}: (\mathbf{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Vect}$ constructed in the previous section associates to each finite colored graph $\Gamma \rightarrow C$ the space of C^∞ (infinitely differentiable) groupoid invariant vector fields $\mathbb{V}(\Gamma)$ and subsequently $S_\Gamma(\mathbb{V}(\Gamma))$ is a subspace of C^∞ vector fields on the Euclidean phase space $\mathbb{P}\Gamma_0$. It is not hard to modify our construction to associate a different class of vector fields, such as linear or analytic. All one has to do is to replace the category \mathbf{Euc} by an appropriate subcategory. Here is a description of what such a replacement entails in the case of linear vector fields.

We replace the category \mathbf{Euc} of Euclidean spaces and C^∞ maps with the category $\mathbf{FinVect}$ of finite dimensional vector spaces over the reals and linear maps. As before, we fix a graph $C = \{C_1 \rightrightarrows C_0\}$ of colors and a phase space function $\mathcal{P}: C_0 \rightarrow \mathbf{FinVect}$. These choices give rise to a phase space functor

$$(3.1.1) \quad \mathbb{P}: (\mathbf{FinSet}/C_0)^{\text{op}} \rightarrow \mathbf{FinVect}.$$

On objects the functor \mathbb{P} is defined as a categorical product (q.v. A.2.4):

$$\mathbb{P}X \equiv \mathbb{P}(X \xrightarrow{\alpha} C_0) := \prod_{x \in X} \mathcal{P}(\alpha(x)).$$

Note that while we use exactly the same notation as before, the product in question is now a

product in $\mathbf{FinVect}$ and not in \mathbf{Euc} . Given a morphism $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \beta & \swarrow \alpha \\ & C_0 & \end{array}$ in the slice category \mathbf{FinSet}/C_0

we obtain

$$\mathbb{P}f: \mathbb{P}Y \rightarrow \mathbb{P}X$$

using the universal property of products (q.v. (2.3.5)). We have already constructed the groupoid of finite colored trees $\mathbf{FinTree}/C$, so we recycle the construction. The definition of the control functor $\mathbf{Ctrl}_{\mathbf{FV}}: \mathbf{FinTree}/C \rightarrow \mathbf{Vect}$ looks almost the same as $\mathbf{Ctrl}: \mathbf{FinTree}/C \rightarrow \mathbf{Euc}$ (q.v. 2.5): for a colored tree $(T \rightarrow C) \in \mathbf{FinTree}/C$ we set

$$\mathbf{Ctrl}_{\mathbf{FV}}(T) := \mathbf{Hom}_{\mathbf{FV}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T).$$

Note that now $\mathbf{Hom}_{\mathbf{FV}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T)$ is the finite dimensional vector space of linear maps from the vector space $\mathbb{P} \text{lv } T$ to the vector space $\mathbb{P} \text{rt } T$ (in fact, $\dim \mathbf{Ctrl}_{\mathbf{FV}}(T) = \dim(\mathbb{P} \text{lv } T) \dim(\mathbb{P} \text{rt } T)$).

Given an isomorphism $\begin{array}{ccc} T & \xrightarrow{\sigma} & T' \\ & \searrow \alpha & \swarrow \beta \\ & C & \end{array}$ of trees over C , we define

$$\mathbf{Ctrl}_{\mathbf{FV}}(\sigma): \mathbf{Ctrl}_{\mathbf{FV}}(T) \rightarrow \mathbf{Ctrl}_{\mathbf{FV}}(T')$$

by the same formula as in (2.5.4):

$$\mathbf{Ctrl}_{\mathbf{FV}}(\sigma)X := X \circ \mathbb{P}(\sigma|_{\text{lv } T}): \mathbb{P} \text{lv } T' \rightarrow \mathbb{P} \text{rt } T = \mathbb{P} \text{rt } T'$$

for any $(X: \mathbb{P} \text{lv } T \rightarrow \mathbb{P} \text{rt } T) \in \mathbf{Ctrl}(T) = \mathbf{Hom}_{\mathbf{FV}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T)$. Note that the image of $\mathbf{Ctrl}_{\mathbf{FV}}$ lands in the subcategory $\mathbf{FinVect}$ of \mathbf{Vect} .

We do not need to change the definitions of input trees (Definition 2.6.1) or of the groupoids $G(\Gamma)$ associated to colored graphs $\Gamma \rightarrow C$ (Definition 2.6.3). We thus define the (finite-dimensional) vector space $\mathbb{V}_{\mathbf{FV}}(\Gamma)$ of virtual groupoid-invariant *linear* vector fields by

$$\mathbb{V}_{\mathbf{FV}}(\Gamma) := \lim(\mathbf{Ctrl}_{\mathbf{FV}}|_{G(\Gamma)}: G(\Gamma) \rightarrow \mathbf{FinVect}).$$

3.2. The relation of $\mathbb{V}_{\text{FV}}(\Gamma)$ to vector fields on the phase space $\mathbb{P}\Gamma_0$.

Given a finite dimensional vector space W the space of linear vector fields on W is (naturally isomorphic to) the space of linear maps $\text{Hom}_{\text{FV}}(W, W)$. Given a finite graph Γ over a graph of colors C and the phase space function $\mathcal{P}: C_0 \rightarrow \text{FinVect}$, the functor \mathbb{P} assigns to the set Γ_0 of vertices of Γ a phase space $\mathbb{P}\Gamma_0$, which is now a finite dimensional vector space. In complete analogy with Section 2.8, there exists a canonical linear map S_Γ from the space of virtual groupoid-invariant vector fields $\mathbb{V}_{\text{FV}}(\Gamma)$ to the vector space $\text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0)$ of linear vector fields on $\mathbb{P}\Gamma_0$. It is defined as follows.

Since $\text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0) = \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \prod_{a \in \Gamma_0} \mathbb{P}\{a\})$ (q.v. Remark A.2.8), the space of linear vector fields on $\mathbb{P}\Gamma_0$ is canonically the product

$$\text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0) = \prod_{a \in \Gamma_0} \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}).$$

To define the map into a product, it is enough to define a map into its factors. Thus we need maps $\mathbb{V}_{\text{FV}}(\Gamma) \rightarrow \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\})$ for each vertex a of the graph Γ .

For each input tree $I(a)$ we have a map of graphs $\xi: I(a) \rightarrow \Gamma$ (q.v. Remark 2.6.2). Therefore we have a map $\xi|_{\text{lv } I(a)}: \text{lv } I(a) \rightarrow \Gamma_0$ of finite sets over C_0 . Applying the phase space functor \mathbb{P} we obtain the maps

$$\mathbb{P}(\xi|_{\text{lv } I(a)}): \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\text{lv } I(a).$$

The pullback by $\mathbb{P}(\xi|_{\text{lv } I(a)})$ gives

$$(\mathbb{P}(\xi|_{\text{lv } I(a)}))^*: \text{Ctrl}(I(a)) \rightarrow \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}).$$

By the universal properties of the product $\chi(\mathbb{P}\Gamma_0)$ there a canonical unique map $S_\Gamma: \mathbb{V}(\Gamma) \rightarrow \text{Hom}_{\text{FinVect}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0)$ making the diagram

$$(3.2.1) \quad \begin{array}{ccc} \mathbb{V}(\Gamma) & \xrightarrow{\quad \exists! S_\Gamma \quad} & \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0) \\ \varpi_a \downarrow & & \downarrow p_a \\ \text{Ctrl}(I(a)) & \xrightarrow{(\mathbb{P}(\xi|_{\text{lv } I(a)}))^*} & \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}) \end{array}$$

commute. Here, as before,

$$\varpi_a: \mathbb{V}_{\text{FV}}\Gamma \rightarrow \text{Ctrl}_{\text{FV}}(\mathbb{P}I(a))$$

and

$$p_a: \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0) = \prod_{a' \in \Gamma_0} \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a'\}) \rightarrow \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\})$$

denote the canonical projections.

3.3. The assignment $\Gamma \mapsto \mathbb{V}_{\text{FV}}\Gamma$ extends to a functor.

Just as for C^∞ vector fields on Euclidean spaces we have

3.3.1. Theorem. *The map \mathbb{V}_{FV} that assigns to each finite graph Γ over C the vector space of $G(\Gamma)$ -invariant virtual linear vector fields on $\mathbb{P}\Gamma_0$ extends to a contravariant functor*

$$\mathbb{V}_{\text{FinVect}}: (\text{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \text{Vect}.$$

Proof. Consider an étale map of graphs over C :

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array} ,$$

31

that is, a morphism in the category $(\mathbf{FinGraph}/C)_{\text{et}}$. Given an arrow $I(a) \xrightarrow{\sigma} I(b)$ in the groupoid $G(\Gamma)$ we have a commuting diagram (2.10.9)

$$\begin{array}{ccc} I(a) & \xrightarrow{\sigma} & I(b) \\ \downarrow \varphi_a & & \downarrow \varphi_b \\ I(\varphi(a)) & \xrightarrow{G(\varphi)\sigma} & I(\varphi(b)) \end{array}$$

in the category of finite trees over C . Applying the control functor Ctrl_{FV} gives us a commuting diagram in the category of vector spaces (q.v. (2.10.11)):

$$(3.3.2) \quad \begin{array}{ccc} \text{Ctrl}_{\text{FV}}(I(a)) & \xrightarrow{\text{Ctrl}_{\text{FV}}(\sigma)} & \text{Ctrl}_{\text{FV}}(I(b)) \\ \downarrow \text{Ctrl}_{\text{FV}}(\varphi_a) & & \downarrow \text{Ctrl}_{\text{FV}}(\varphi_b) \\ \text{Ctrl}_{\text{FV}}(I(\phi(a))) & \xrightarrow{\text{Ctrl}_{\text{FV}}(G(\varphi)\sigma)} & \text{Ctrl}_{\text{FV}}(I(\phi(b))) \end{array}$$

We now could mimic the rest of the proof of 2.10.6. However, we choose to proceed in a more functorial fashion. Diagram (3.3.2) gives us the 2-commutative diagram

$$\begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ & \searrow \phi & \swarrow \\ \text{Ctrl}_{\text{FV}|G(\Gamma)} & & \text{Ctrl}_{\text{FV}|G(\Gamma')} \\ & \searrow & \swarrow \\ & \text{Vect} & \end{array},$$

q.v. Remark 2.10.14. This 2-commuting triangle is a morphism in $\mathbf{Cat}/\mathbf{Vect}$ (see 6.1.2). It is not hard to check that the assignment

$$\mathcal{G}: (\mathbf{FinGraph}/C)_{\text{et}} \rightarrow \mathbf{Cat}/\mathbf{Vect}, \quad \begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array} \mapsto \begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ & \searrow \phi & \swarrow \\ \text{Ctrl}_{\text{FV}|G(\Gamma)} & & \text{Ctrl}_{\text{FV}|G(\Gamma')} \\ & \searrow & \swarrow \\ & \text{Vect} & \end{array}$$

preserves the composition of morphisms and thus is a functor. Composing the functor \mathcal{G} with the contravariant functor $L: \mathbf{Cat}/\mathbf{Vect}^{\text{op}} \rightarrow \mathbf{Vect}$ (see 6.1.3) gives us the desired contravariant functor

$$\mathbb{V}_{\text{FV}} = \mathcal{G} \circ L: (\mathbf{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \mathbf{Vect}.$$

□

3.4. The combinatorial category \mathcal{V}^{FV} of linear dynamical systems.

Associated to the functor \mathbb{V}_{FV} we have the category of elements \mathcal{V}^{FV} and a functor $\mathcal{V}^{\text{FV}} \rightarrow (\mathbf{FinGraph}/C)_{\text{et}}^{\text{op}}$. As in the case of \mathcal{V}^{Euc} the objects of \mathcal{V}^{FV} are pairs of the form (Γ, v) with Γ a finite graph over C and $v \in \mathbb{V}_{\text{FV}}(\Gamma)$. The morphisms in \mathcal{V}^{FV} are the tuples of the form $(h, (v, \Gamma), (v', \Gamma'))$ with $h: \Gamma' \rightarrow \Gamma$ a morphism in $\mathbf{FinGraph}/C$ and $\mathbb{V}_{\text{FinGraph}}(h)v = v'$.

As in the case of C^∞ vector fields on Euclidean spaces the maps

$$\{S_\Gamma: \mathbb{V}_{\text{FV}}(\Gamma) \rightarrow \text{Hom}_{\text{FV}}(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0)\}_{\Gamma \in (\mathbf{FinGraph}/C)_0}$$

assemble into a functor S from the category of elements \mathcal{V}^{FV} to the category \mathbf{LinDyn} of *linear* dynamical systems. The key technical result in proving that S is a functor is the analogue of Theorem 2.11.2:

3.4.1. Theorem. *Let $w \in \mathbb{V}_{\text{FV}}(\Gamma')$ be a virtual linear dynamical system on $\mathbb{P}(\Gamma)$. For each étale map of graphs*

$$\varphi: \Gamma \rightarrow \Gamma'$$

the diagram

$$(3.4.2) \quad \begin{array}{ccc} \mathbb{P}(\Gamma'_0) & \xrightarrow{\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \\ S_{\Gamma'}(w) \uparrow & & \uparrow S_{\Gamma}(\mathbb{V}(\varphi)w) \\ \mathbb{P}(\Gamma'_0) & \xrightarrow{\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \end{array}$$

commutes. In other words,

$$\mathbb{P}(\varphi): (\mathbb{P}(\Gamma'), S_{\Gamma'}(w)) \rightarrow (\mathbb{P}(\Gamma), S_{\Gamma}(\mathbb{V}(\varphi)w))$$

is a map of linear dynamical systems.

Since the proof of 3.4.1 is completely analogous to the proof of 2.11.2 we omit it. We conclude the section by stating the analogue of Theorem 2.12.3:

3.4.3. Theorem. *Let \mathcal{V}^{FV} denote the category of virtual groupoid-invariant dynamical systems constructed above, $\pi: \mathcal{V}^{\text{FV}} \rightarrow (\text{FinGraph}/C)_{\text{et}}^{\text{op}}$ denote the canonical projection, $U: \text{LinDyn} \rightarrow \text{FinVect}$ the forgetful functor, and \mathbb{P} the phase space functor extended to finite graphs (q.v. 2.3.14). There exists a functor $S: \mathcal{V}^{\text{FV}} \rightarrow \text{LinDyn}$ making the diagram*

$$(3.4.4) \quad \begin{array}{ccc} \mathcal{V}^{\text{FV}} & \xrightarrow{S} & \text{LinDyn} \\ \pi \downarrow & & \downarrow U \\ (\text{FinGraph}/C)_{\text{et}}^{\text{op}} & \xrightarrow{\mathbb{P}} & \text{FinVect} \end{array}$$

commute. On objects

$$S(v, \Gamma) = (\mathbb{P}\Gamma_0, S_{\Gamma}(v));$$

on arrows

$$S(h, (v, \Gamma), (v', \Gamma')) = (\mathbb{P}\Gamma_0, S_{\Gamma}(v)) \xrightarrow{\mathbb{P}h} (\mathbb{P}\Gamma', S_{\Gamma'}(v)).$$

3.4.5. Remark. If $\varphi: \Gamma \rightarrow \Gamma'$ is an isomorphism of graphs, then $\mathbb{P}\varphi$ and $\mathbb{V}\varphi$ are isomorphisms of vector spaces, since \mathbb{P} and \mathbb{V} are functors. Therefore (3.4.2) is a similarity of linear transformations. It is not hard to see, however, that the only similarities we can obtain in this matter are permutations of blocks of identities (see (2.3.8)). In any case, $S_{\Gamma}(w)$ and $S_{\Gamma'}(\mathbb{V}(\varphi)w)$ have precisely the same spectrum when thought of as linear transformations. If φ is not an isomorphism, then even though (3.4.2) is in the form of a similarity transformation, the relationship between the spectra of $S_{\Gamma}(w)$ and $S_{\Gamma'}(\mathbb{V}(\varphi)w)$ is a complicated question, q.v. [61, 62]. A partial resolution of this question will be useful for an application which we will present in [54].

4. GROUPOID INVARIANT VECTOR FIELDS ON MANIFOLDS

We now extend the results of Section 2 from the category **Euc** of Euclidean spaces to the category **Man** of (Hausdorff paracompact finite dimensional C^∞) manifolds. Since the constructions in Section 2 are functorial and, in particular, do not use coordinates, such an extension is not difficult. However, we do need to modify our definition of the control functor **Ctrl**.

4.1. Control systems functor.

Recall the construction

$$\text{Ctrl}: \text{FinTree}/C \rightarrow \text{Vect}$$

in the case of Euclidean spaces. First, a choice of a phase space function $\mathcal{P}: C_0 \rightarrow \text{Euc}$ gives rise to a phase space functor

$$\mathbb{P}: (\text{FinSet}/C_0)^{\text{op}} \rightarrow \text{Euc}.$$

Then given a finite tree T over a graph C of colors we defined the space of control systems $\text{Ctrl}(T)$ by

$$\text{Ctrl}(T) := \text{Hom}_{\text{Euc}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T),$$

which is canonically a subspace of the space of control systems $\text{CT}(\mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow \mathbb{P} \text{rt } T)$ (q.v. 2.1.6 and 2.5.2).

Suppose we mimic this definition in the case of manifolds. As remarked in Section 2.3, just as in the case of Euclidean spaces, a choice of a phase space function $\mathcal{P}: C_0 \rightarrow \text{Man}$ gives rise to a phase space functor

$$\mathbb{P}: (\text{FinSet}/C_0)^{\text{op}} \rightarrow \text{Man}.$$

On objects

$$\mathbb{P}X \equiv \mathbb{P}(X \xrightarrow{\alpha} C_0) := \prod_{x \in X} \mathcal{P}(\alpha(x)),$$

where now the product is in the category Man . Since Man has finite products, \mathbb{P} is well-defined on objects. Implicit in this definition is a *choice* of a manifold $\prod_{x \in X} \mathcal{P}(\alpha(x))$ and a family of submersions $\{p_x: \prod_{x' \in X} \mathcal{P}(\alpha(x')) \rightarrow \mathcal{P}(\alpha(x))\}_{x \in X}$. By the universal property of products for any

morphism $\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \beta & \swarrow \alpha \\ & C_0 & \end{array}$ in the slice category FinSet/C_0 , we get a map of manifolds $\mathbb{P}f: \mathbb{P}X \rightarrow \mathbb{P}Y$ satisfying

$$\begin{array}{ccc} \mathbb{P}X & \xrightarrow{\quad \mathbb{P}f \quad} & \mathbb{P}Y \\ p_{f(y)} \downarrow & & \downarrow q_y \\ \mathcal{P}(\alpha(f(y))) & \xlongequal{\quad} & \mathcal{P}(\beta(y)) \end{array},$$

q.v. 2.3.12. However, the embedding (2.1.7)

$$\text{Hom}_{\text{Euc}}(V, W) \hookrightarrow \text{CT}(V \times W \rightarrow W), \quad f \mapsto (pr_2, F)$$

has no analogue in the category of manifolds. Indeed what makes the embedding work is that the tangent space TW of a Euclidean space W is canonically isomorphic to $W \times W$: $TW \xrightarrow{\cong} W \times W$. Thus defining the control functor Ctrl_{Man} on trees by

$$\text{Ctrl}_{\text{Man}}(T) = \text{Hom}_{\text{Man}}(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T) = C^\infty(\mathbb{P} \text{lv } T, \mathbb{P} \text{rt } T)$$

does not make sense for a general manifold.⁶ However, given a finite tree T over C and a phase space functor $\mathbb{P}: (\mathbf{FinSet}/C)^{\text{op}} \rightarrow \mathbf{Man}$ we do have a canonical projection

$$pr_2: \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow \mathbb{P} \text{rt } T,$$

which is a surjective submersion. Consequently we can talk about the space $\mathbf{CT}(pr_2: \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow \mathbb{P} \text{rt } T)$ of all control systems supported on pr_2 , see 2.1.5. Therefore,

4.1.1. Definition (The space of control systems associated to a tree). Given a finite tree T over a graph C of colors and a phase space functor $\mathbb{P}: (\mathbf{FinSet}/C)^{\text{op}} \rightarrow \mathbf{Man}$, we define the infinite dimensional vector space of *control systems associated to the tree $T \rightarrow C$* by

$$\begin{aligned} \mathbf{Ctrl}_{\mathbf{Man}}(T) &:= \\ &\{F: \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow T \mathbb{P} \text{rt } T \mid F(u, m) \in T_m(\mathbb{P} \text{rt } T) \text{ for all } (u, m) \in \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T\} \\ &\equiv \mathbf{CT}(pr_2: \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \rightarrow \mathbb{P} \text{rt } T). \end{aligned}$$

4.1.2. Remark. If the space of leaves $\text{lv } T$ is empty (so that T consists of a single vertex) then $\mathbb{P} \text{lv } T$ is a point and $\mathbf{Ctrl}_{\mathbf{Man}}(T)$ is simply the space of all vector fields on the phase space $\mathbb{P} \text{rt } T$ (q.v. Remark 2.1.4).

The assignment $T \mapsto \mathbf{Ctrl}_{\mathbf{Man}}(T)$ extends to a functor $\mathbf{Ctrl}_{\mathbf{Man}}: \mathbf{FinTree}/C \rightarrow \mathbf{Vect}$, from the category of finite trees over C to the category of infinite dimensional vector spaces as follows. Let

$$\begin{array}{ccc} T & \xrightarrow{\sigma} & T' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array}$$

be an isomorphism of trees. Then we have two isomorphisms of finite sets over C_0 :

$$\sigma|_{\text{lv } T}: \text{lv } T \rightarrow \text{lv } T' \quad \text{and} \quad \sigma|_{\text{rt } T}: \text{rt } T \rightarrow \text{rt } T'.$$

Applying the phase space functor \mathbb{P} we get two diffeomorphisms of manifolds

$$\mathbb{P}(\sigma|_{\text{lv } T}): \mathbb{P} \text{lv } T' \rightarrow \mathbb{P} \text{lv } T, \quad \mathbb{P}(\sigma|_{\text{rt } T}): \mathbb{P} \text{rt } T' \rightarrow \mathbb{P} \text{rt } T.$$

Note that the second diffeomorphism is the identity map, see 2.3.9. In other words the square

$$\begin{array}{ccc} \mathbb{P} \text{lv } T' \times \mathbb{P} \text{rt } T' & \xrightarrow{\mathbb{P}(\sigma|_{\text{lv } T}) \times \mathbb{P}(\sigma|_{\text{rt } T})} & \mathbb{P} \text{lv } T \times \mathbb{P} \text{rt } T \\ \downarrow pr_2 & & \downarrow pr_2 \\ \mathbb{P} \text{rt } T' & \xrightarrow{\mathbb{P}(\sigma|_{\text{rt } T}) = id_{\mathbb{P}(\text{rt } T)}} & \mathbb{P} \text{rt } T \end{array}$$

commutes. Consequently we get a map

$$(4.1.3) \quad \mathbf{Ctrl}_{\mathbf{Man}}(\sigma): \mathbf{Ctrl}_{\mathbf{Man}}(T) \rightarrow \mathbf{Ctrl}_{\mathbf{Man}}(T')$$

which is the pull-back by $\mathbb{P}(\sigma|_{\text{lv } T}) \times \mathbb{P}(\sigma|_{\text{rt } T}) = \mathbb{P}(\sigma|_{\text{lv } T}) \times id_{\mathbb{P} \text{rt } T}$ composed with the pushforward by $d(\mathbb{P}\sigma|_{\text{rt } T})$:

$$(4.1.4) \quad \mathbf{Ctrl}_{\mathbf{Man}}(\sigma)X := d(\mathbb{P}\sigma|_{\text{rt } T}) \circ X \circ (\mathbb{P}(\sigma|_{\text{lv } T}) \times \mathbb{P}(\sigma|_{\text{rt } T})): \mathbb{P} \text{lv } T' \times \mathbb{P} \text{rt } T' \rightarrow T \mathbb{P} \text{rt } T'$$

⁶One can try and salvage this approach by restricting the definition of the \mathbf{Ctrl} functor to a subcategory of canonically parallelizable manifolds (e.g., the category of abelian Lie groups and smooth maps) and setting

$$\mathbf{Ctrl}_{\mathbf{Man}}(T) := C^\infty(\mathbb{P} \text{lv } T, T_x(\mathbb{P} \text{rt } T))$$

for some arbitrary point $x \in \mathbb{P} \text{rt } T$. Any trivialization of the fiber bundle $T(\mathbb{P} \text{rt } T) \simeq \mathbb{P} \text{rt } T \times T_x(\mathbb{P} \text{rt } T)$ would then give an embedding $C^\infty(V, W) \hookrightarrow \mathbf{CT}(V \times W \rightarrow W)$. We do not pursue this idea further here, but see Golubitsky, Josić, and Shea-Brown [50] for a beautiful application.

for any $(X: \mathbb{P}lv T \times \mathbb{P}rt T \rightarrow T\mathbb{P}rt T) \in \mathbf{Ctrl}_{\mathbf{Man}}(T)$. Once again we leave it to the reader to check that $\mathbf{Ctrl}_{\mathbf{Man}}$ preserves the composition of morphisms, that is, that $\mathbf{Ctrl}_{\mathbf{Man}}$ is a functor.

4.1.5. Remark. Suppose the phase space function $\mathcal{P}: C_0 \rightarrow \mathbf{Man}$ happens to take its values in the Euclidean spaces. The functors $\mathbf{Ctrl} = \mathbf{Ctrl}_{\mathbf{Euc}}$ and $\mathbf{Ctrl}_{\mathbf{Man}}$ are quite different:

Given a tree T , we defined $\mathbf{Ctrl}_{\mathbf{Euc}}(T) = C^\infty(\mathbb{P}lv T, \mathbb{P}rt T)$, and identified it with a space of control system by mapping a function $f \in \mathbf{Ctrl}_{\mathbf{Euc}}(T)$ to the control system $F: \mathbb{P}lv T \times \mathbb{P}rt T \rightarrow T\mathbb{P}rt T$ by $F(v, w) = (w, f(v))$. In particular F so defined does not depend on the points of the Euclidean space $\mathbb{P}rt T$. On the other hand we defined $\mathbf{Ctrl}_{\mathbf{Man}}(T)$ as the space of all smooth maps $F: \mathbb{P}lv T \times \mathbb{P}rt T \rightarrow T\mathbb{P}rt T$ with $F(u, m) \in T_m(\mathbb{P}rt T)$. So such a control system F does in general depend on the points of the Euclidean space $\mathbb{P}rt T$.

As a consequence the spaces of virtual groupoid invariant vector fields $\mathbb{V}(\Gamma)$ and $\mathbb{V}_{\mathbf{Man}}(\Gamma)$ and their respective images $S_\Gamma(\mathbb{V}(\Gamma))$, $S_\Gamma(\mathbb{V}_{\mathbf{Man}}(\Gamma))$ are different even if the phase space function \mathcal{P} is the same (the official definitions of $\mathbb{V}_{\mathbf{Man}}(\Gamma)$ and $S_\Gamma: \mathbb{V}_{\mathbf{Man}}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0)$ are given in the next two subsections). For example, for the graph



(this is a graph colored by a graph C with one vertex and *two* distinct edges) the vector fields in $S_\Gamma(\mathbb{V}(\Gamma))$ are of the form

$$\dot{x}_1 = f(x_2), \quad \dot{x}_2 = g(x_1),$$

whereas the vector fields in $S_\Gamma(\mathbb{V}_{\mathbf{Man}}(\Gamma))$ are of the form

$$\dot{x}_1 = k(x_2, x_1), \quad \dot{x}_2 = h(x_1, x_2).$$

4.2. Virtual groupoid invariant vector fields on manifolds.

In complete analogy with Definition 2.7.2 we now have:

4.2.1. Definition (Virtual groupoid invariant vector fields on manifolds). Fix the graph of colors $C = \{C_1 \rightrightarrows C_0\}$ and a phase space function $\mathcal{P}: C_0 \rightarrow \mathbf{Man}$. Given a graph Γ over C we define the vector space of *virtual groupoid invariant vector fields* $\mathbb{V}_{\mathbf{Man}}(\Gamma)$ to be the limit of the restriction of the functor $\mathbf{Ctrl}_{\mathbf{Man}}$ to the groupoid $G(\Gamma)$:

$$\mathbb{V}_{\mathbf{Man}}(\Gamma) := \lim(\mathbf{Ctrl}_{\mathbf{Man}}|_{G(\Gamma)}: G(\Gamma) \rightarrow \mathbf{Vect}).$$

Note that since $\mathbb{V}_{\mathbf{Man}}(\Gamma)$ is a limit, it comes with a family of the canonical projections

$$\{\varpi_a: \mathbb{V}_{\mathbf{Man}}(\Gamma) \rightarrow \mathbf{Ctrl}_{\mathbf{Man}}(I(a))\}_{a \in \Gamma_0}.$$

Immediately we have the analogue of Remark 2.7.3:

4.2.2. Remark. The vector space $\mathbb{V}_{\mathbf{Man}}(\Gamma)$ has the following concrete description. Consider the product $\prod_{a \in \Gamma_0} \mathbf{Ctrl}_{\mathbf{Man}}(I(a))$ with its canonical projections $p_b: \prod_{a \in \Gamma_0} \mathbf{Ctrl}_{\mathbf{Man}}(I(a)) \rightarrow \mathbf{Ctrl}_{\mathbf{Man}}(I(b))$, $b \in \Gamma_0$. For a vector $X \in \prod_{a \in \Gamma_0} \mathbf{Ctrl}_{\mathbf{Man}}(I(a))$, denote the a -th component $p_a(X)$ of X by X_a . Then

$$\mathbb{V}_{\mathbf{Man}}(\Gamma) = \{X \in \prod_{a \in \Gamma_0} \mathbf{Ctrl}_{\mathbf{Man}}(I(a)) \mid \mathbf{Ctrl}_{\mathbf{Man}}(\sigma)X_a = X_b \text{ for all arrows } I(a) \xrightarrow{\sigma} I(b) \in G(\Gamma)\}$$

with the canonical projections $\varpi_a: \mathbb{V}_{\mathbf{Man}}(\Gamma) \rightarrow \mathbf{Ctrl}_{\mathbf{Man}}(I(a))$ given by restrictions $p_a|_{\mathbb{V}_{\mathbf{Man}}(\Gamma)}$.

4.3. The relation of $\mathbb{V}_{\text{Man}}(\Gamma)$ to vector fields on the phase space $\mathbb{P}\Gamma_0$.

Given a product $M = M_1 \times \dots \times M_k$ of manifolds, the space of C^∞ vector fields $\chi(M)$ is naturally a product of control systems

$$\chi(M_1 \times \dots \times M_k) = \prod_{i=1}^k \text{CT}(p_i: M_1 \times \dots \times M_k \rightarrow M_i).$$

Therefore, given a phase space functor $\mathbb{P}: (\text{FinGraph}/C)^{\text{op}} \rightarrow \text{Man}$ for each graph Γ over the graph of colors C we have

$$\chi(\mathbb{P}\Gamma_0) = \prod_{a \in \Gamma_0} \text{CT}(\mathbb{P}\kappa_a: \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\})$$

where $\kappa_a: \{a\} \hookrightarrow \Gamma_0$ is the canonical inclusion of a vertex $\{a\}$ into the set of vertices of the graph Γ . Note that the canonical projections

$$(4.3.1) \quad \pi_a: \prod_{a' \in \Gamma_0} \text{CT}(\mathbb{P}\kappa_{a'}: \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a'\}) \rightarrow \text{CT}(\mathbb{P}\kappa_a: \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\})$$

are given by push-forwards:

$$(4.3.2) \quad \pi_a(X) = d\mathbb{P}\kappa_a \circ X \equiv (d\mathbb{P}\kappa_a)_* X.$$

Since $\mathbb{V}_{\text{Man}}(\Gamma)$ is a limit, for each node a we also have a canonical projection

$$\varpi_a: \mathbb{V}_{\text{Man}}\Gamma \rightarrow \text{Ctrl}_{\text{Man}}(I(a)).$$

Recall that by definition $\text{Ctrl}_{\text{Man}}(I(a))$ is the space of all control systems supported by the surjective submersion $\mathbb{P} \text{lv } I(a) \times \mathbb{P} \text{rt } I(a) \xrightarrow{pr_2} \mathbb{P} \text{rt } I(a)$:

$$\text{Ctrl}_{\text{Man}}(I(a)) = \text{CT}(pr_2: \mathbb{P} \text{lv } I(a) \times \mathbb{P} \text{rt } I(a) \rightarrow \mathbb{P} \text{rt } I(a)).$$

Since the set $I(a)_0$ of nodes of the input tree $I(a)$ is simply the union of the leaves $\text{lv } I(a)$ and the root $\text{rt } I(a) = \{a\}$ we have

$$(4.3.3) \quad \text{Ctrl}_{\text{Man}}(I(a)) = \text{CT}(\mathbb{P}I(a)_0 \rightarrow \mathbb{P}\{a\}).$$

Recall that for each input tree $I(a)$ of a graph Γ we have a map of graphs $\xi: I(a) \rightarrow \Gamma$ (q.v. 2.6.2), hence a map

$$\xi: I(a)_0 \rightarrow \Gamma_0$$

of finite sets over the set of colors C_0 . Applying the phase space functor \mathbb{P} we obtain canonical maps of manifolds

$$\mathbb{P}(\xi): \mathbb{P}\Gamma_0 \rightarrow \mathbb{P}I(a)_0.$$

The pullback by $\mathbb{P}(\xi)$ gives

$$\mathbb{P}(\xi)^*: \text{Ctrl}_{\text{Man}}(I(a)) = \text{CT}(\mathbb{P}I(a)_0 \rightarrow \mathbb{P}\{a\}) \rightarrow \text{CT}(\mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\}).$$

By the universal properties of the product

$$\{\pi_a: \chi(\mathbb{P}\Gamma_0) \rightarrow \text{CT}(\mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\})\}_{a \in \Gamma_0}$$

there a unique canonical map $S_\Gamma: \mathbb{V}_{\text{Man}}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0)$ making the diagram

$$(4.3.4) \quad \begin{array}{ccc} \mathbb{V}_{\text{Man}}\Gamma & \overset{\exists! S_\Gamma}{\dashrightarrow} & \chi(\mathbb{P}\Gamma_0) \\ \varpi_a \downarrow & & \downarrow \pi_a \\ \text{Ctrl}_{\text{Man}}(I(a)) & \xrightarrow{\mathbb{P}(\xi)^*} & \text{CT}(\mathbb{P}\Gamma_0, \mathbb{P}\{a\}) \end{array}$$

commute.

4.4. The assignment $\Gamma \mapsto \mathbb{V}_{\text{Man}}(\Gamma)$ extends to a functor.

As in the case of groupoid invariant smooth vector fields on Euclidean spaces and in the case of groupoid invariant linear vector fields we have

4.4.1. Theorem. *The map \mathbb{V}_{Man} that assigns to each finite graph Γ over C the vector space of $G(\Gamma)$ invariant virtual vector fields on $\mathbb{P}\Gamma_0$ extends to a contravariant functor*

$$\mathbb{V}_{\text{Man}} : (\text{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \text{Vect}.$$

Proof. We mimic the proof of Theorem 3.3.1: Consider an étale map of graphs over C :

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array},$$

that is, a morphism in the category $(\text{FinGraph}/C)_{\text{et}}$. Given an arrow $I(a) \xrightarrow{\sigma} I(b)$ in the groupoid $G(\Gamma)$ we have a commuting diagram (2.10.9)

$$\begin{array}{ccc} I(a) & \xrightarrow{\sigma} & I(b) \\ \downarrow \varphi_a & & \downarrow \varphi_b \\ I(\varphi(a)) & \xrightarrow{G(\varphi)\sigma} & I(\varphi(b)) \end{array}$$

in the category of finite trees over C . Applying the control functor Ctrl_{Man} gives us a commuting diagram in the category of vector spaces (q.v. (2.10.11) and (3.3.2)):

$$(4.4.2) \quad \begin{array}{ccc} \text{Ctrl}_{\text{Man}}(I(a)) & \xrightarrow{\text{Ctrl}_{\text{Man}}(\sigma)} & \text{Ctrl}_{\text{Man}}(I(b)) \\ \text{Ctrl}_{\text{Man}}(\varphi_a) \downarrow & & \downarrow \text{Ctrl}_{\text{Man}}(\varphi_b) \\ \text{Ctrl}_{\text{Man}}(I(\varphi(a))) & \xrightarrow{\text{Ctrl}_{\text{Man}}(G(\varphi)\sigma)} & \text{Ctrl}_{\text{Man}}(I(\varphi(b))), \end{array}$$

which, in turn, gives us the 2-commutative diagram

$$\begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ \downarrow \text{Ctrl}_{\text{Man}}|_{G(\Gamma)} & \searrow \phi & \swarrow \text{Ctrl}_{\text{Man}}|_{G(\Gamma')} \\ & \text{Vect} & \end{array},$$

q.v. Remark 2.10.14. This 2-commuting triangle is a morphism in Cat/Vect (see 6.1.2). It is not hard to check that the assignment

$$\mathcal{G} : (\text{FinGraph}/C)_{\text{et}} \rightarrow \text{Cat}/\text{Vect}, \quad \begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & \Gamma' \\ \alpha \searrow & & \swarrow \beta \\ & C & \end{array} \mapsto \begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ \downarrow \text{Ctrl}_{\text{Man}}|_{G(\Gamma)} & \searrow \phi & \swarrow \text{Ctrl}_{\text{Man}}|_{G(\Gamma')} \\ & \text{Vect} & \end{array}$$

preserves the composition of morphisms and thus is a functor. Composing the functor \mathcal{G} with the contravariant functor $L : \text{Cat}/\text{Vect}^{\text{op}} \rightarrow \text{Vect}$ (see 6.1.3) gives us the desired contravariant functor

$$\mathbb{V}_{\text{Man}} = \mathcal{G} \circ L : (\text{FinGraph}/C)_{\text{et}}^{\text{op}} \rightarrow \text{Vect}.$$

□

4.4.3. **Remark.** Unraveling the definition of \mathbb{V}_{Man} we see that for any étale map of graphs $\varphi: \Gamma \rightarrow \Gamma'$, any node a of Γ and any virtual groupoid invariant vector field $w \in \mathbb{V}_{\text{Man}}(\Gamma')$ we have

$$(4.4.4) \quad \varpi_a(\mathbb{V}(\varphi)w) = \text{Ctrl}_{\text{Man}}(\varphi_a)^{-1}(\varpi_{\varphi(a)}w)$$

Compare with (2.11.17) and (2.10.12).

4.5. The combinatorial category \mathcal{V}^{Man} of dynamical systems on manifolds.

Associated to the functor \mathbb{V}_{Man} we have the category of elements \mathcal{V}^{Man} and a functor $\mathcal{V}^{\text{Man}} \rightarrow (\text{FinGraph}/C)_{\text{et}}^{\text{op}}$. As in the case of \mathcal{V}^{Euc} the objects of \mathcal{V}^{Man} are pairs of the form (Γ, v) with Γ a finite graph over C and $v \in \mathbb{V}_{\text{Man}}(\Gamma)$. The morphism in \mathcal{V}^{Man} are the tuples of the form $(h, (v, \Gamma), (v', \Gamma'))$ with $h: \Gamma' \rightarrow \Gamma$ a morphism in $\text{FinGraph}/C$ and $\mathbb{V}_{\text{FinGraph}}(h)v = v'$. As in the case of C^∞ vector fields on Euclidean spaces the maps

$$\{S_\Gamma: \mathbb{V}_{\text{Man}}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0, \mathbb{P}\Gamma_0)\}_{\Gamma \in (\text{FinGraph}/C)_0}$$

assemble into a functor S from the category of elements \mathcal{V}^{Man} to the category Dynamical Systems of dynamical systems. The key technical result in proving that S is a functor is the analogue of Theorem 2.11.2:

4.5.1. **Theorem.** *Let $w \in \mathbb{V}_{\text{Man}}(\Gamma')$ be a virtual dynamical system on $\mathbb{P}\Gamma_0$. For each étale map of graphs*

$$\varphi: \Gamma \rightarrow \Gamma'$$

the diagram

$$(4.5.2) \quad \begin{array}{ccc} \mathbb{P}(\Gamma'_0) & \xrightarrow{d\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \\ S_{\Gamma'}(w) \uparrow & & \uparrow S_\Gamma(\mathbb{V}(\varphi)w) \\ \mathbb{P}(\Gamma'_0) & \xrightarrow{\mathbb{P}\varphi} & \mathbb{P}(\Gamma_0) \end{array}$$

commutes: for any point $x \in \mathbb{P}(\Gamma'_0)$

$$(4.5.3) \quad (d\mathbb{P}\varphi)_x(S_{\Gamma'}(w)(x)) = S_\Gamma(\mathbb{V}(\varphi)w)(\mathbb{P}\varphi(x)).$$

In other words,

$$\mathbb{P}(\varphi): (\mathbb{P}(\Gamma'), S_{\Gamma'}(w)) \rightarrow (\mathbb{P}(\Gamma), S_\Gamma(\mathbb{V}(\varphi)w))$$

is a map of dynamical systems.

Proof. We modify the proof of Theorem 2.11.2 to take into account the difference between the functors $\text{Ctrl} = \text{Ctrl}_{\text{Euc}}$ and Ctrl_{Man} . As before, for each node a of a graph Γ over the graph of colors C we have a canonical map $\xi: I(a) \rightarrow \Gamma$ from the input tree $I(a)$ to the graph Γ . As above we use the same letter ξ for all nodes of Γ and for all graphs over C suppressing both dependencies.

If $\varphi: \Gamma \rightarrow \Gamma'$ is an étale map of graphs over C , then by definition of “étale” for each node a of Γ we have an induced isomorphism of graphs over C :

$$\varphi_a: I(a) \rightarrow I(\varphi(a)).$$

Moreover, the diagram of graphs over C

$$(4.5.4) \quad \begin{array}{ccc} I(a) & \xrightarrow{\xi} & \Gamma \\ \downarrow \varphi_a & & \downarrow \varphi \\ I(\varphi(a)) & \xrightarrow{\xi} & \Gamma' \end{array}$$

commutes. Hence we have a commuting diagram

$$(4.5.5) \quad \begin{array}{ccc} \mathbb{P}I(a)_0 & \xleftarrow{\mathbb{P}\xi} & \mathbb{P}\Gamma_0 \\ \mathbb{P}\varphi_a \uparrow & & \uparrow \mathbb{P}\varphi \\ \mathbb{P}I(\varphi(a))_0 & \xleftarrow{\mathbb{P}\xi} & \mathbb{P}\Gamma'_0 \end{array}$$

in our category \mathbf{Man} of phase spaces. As before let

$$\kappa_a : \{a\} \hookrightarrow \Gamma_0$$

denote the canonical inclusion in the category of finite sets over C_0 . We then have a commuting diagram in \mathbf{FinSet}/C_0

$$(4.5.6) \quad \begin{array}{ccc} \{a\} & \xrightarrow{\kappa_a} & \Gamma_0 \\ \varphi_a|_{\{a\}} = \varphi|_{\{a\}} \downarrow & & \downarrow \varphi \\ \{\varphi(a)\} & \xrightarrow{\kappa_{\varphi(a)}} & \Gamma'_0, \end{array}$$

hence a commuting diagram in \mathbf{Man} :

$$(4.5.7) \quad \begin{array}{ccc} \mathbb{P}\{a\} & \xleftarrow{\mathbb{P}\kappa_a} & \mathbb{P}\Gamma_0 \\ \mathbb{P}(\varphi|_{\{a\}}) \uparrow & & \uparrow \mathbb{P}\varphi \\ \mathbb{P}\{\varphi(a)\} & \xleftarrow{\mathbb{P}\kappa_{\varphi(a)}} & \mathbb{P}\Gamma'_0, \end{array}$$

Differentiating (4.5.7) we get the commuting diagram

$$(4.5.8) \quad \begin{array}{ccc} T\mathbb{P}\{a\} & \xleftarrow{d\mathbb{P}\kappa_a} & T\mathbb{P}\Gamma_0 \\ d\mathbb{P}(\varphi|_{\{a\}}) \uparrow & & \uparrow d\mathbb{P}\varphi \\ T\mathbb{P}\{\varphi(a)\} & \xleftarrow{d\mathbb{P}\kappa_{\varphi(a)}} & T\mathbb{P}\Gamma'_0, \end{array}$$

of tangent bundles. Recall that the projection

$$\pi_a : \chi(\mathbb{P}\Gamma_0) \rightarrow \mathbf{CT}(\mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\})$$

is given by

$$\pi_a(X) = d(\mathbb{P}\kappa_a) \circ X,$$

(q.v. (4.3.2)). By definition of the maps S_Γ (q.v. (2.8.2)) we have

$$(4.5.9) \quad d\mathbb{P}\kappa_a \circ S_\Gamma(v) = \varpi_a(v) \circ \mathbb{P}(\xi) \quad \text{for any } a \in \Gamma_0 \text{ and any } v \in \mathbb{V}(\Gamma).$$

Similarly for Γ' we have

$$(4.5.10) \quad d\mathbb{P}\kappa_{\varphi(a)} \circ S_{\Gamma'}(v') = \varpi_{\varphi(a)}(v') \circ \mathbb{P}(\xi) \quad \text{for any } a \in \Gamma_0 \text{ and any } v' \in \mathbb{V}(\Gamma').$$

Recall that the definition of $\mathbb{V}_{\mathbf{Man}}(\varphi)$ implies that

$$(4.5.11) \quad \varpi_a(\mathbb{V}(\varphi)w) = \mathbf{Ctrl}(\varphi_a)^{-1}(\varpi_{\varphi(a)}w)$$

(q.v. Remark 4.4.3). Since the set of nodes of the input tree $I(a)$ is the union of its root and its leaves (4.1.4) and (4.5.11) imply that

$$(4.5.12) \quad \varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi_a = d(\mathbb{P}\varphi|_{\{a\}}) \circ \varpi_{\varphi(a)}w$$

for any node a of Γ and any $w \in \mathbb{V}(\Gamma')$.

Since the vector space of vector fields $\chi(\mathbb{P}\Gamma_0)$ is the product

$$\chi(\mathbb{P}\Gamma_0) = \prod_{a \in \Gamma_0} CT(\mathbb{P}\Gamma_0 \rightarrow \mathbb{P}\{a\})$$

the diagram (4.5.2) commutes if and only if

$$\pi_a(S_\Gamma(\mathbb{V}_{\text{Man}}(\varphi)w) \circ \mathbb{P}\varphi) = \pi_a(d\mathbb{P}\varphi \circ S_{\Gamma'}(w))$$

for every $a \in \Gamma_0$. We now compute:

$$\begin{aligned} \pi_a(S_\Gamma(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi) &= d\mathbb{P}\kappa_a \circ S_\Gamma(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi \\ &= (\varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}\xi) \circ \mathbb{P}\varphi, \quad \text{by (4.5.9)} \\ &= \varpi_a(\mathbb{V}(\varphi)w) \circ (\mathbb{P}\xi \circ \mathbb{P}\varphi) \\ &= \varpi_a(\mathbb{V}(\varphi)w) \circ \mathbb{P}\varphi_a \circ \mathbb{P}\xi \quad \text{since (4.5.5) commutes} \\ &= d(\mathbb{P}\varphi|_{\{a\}}) \circ \varpi_{\varphi(a)}w \circ \mathbb{P}\xi, \quad \text{by (4.5.12)} \\ &= d\mathbb{P}(\varphi|_{\{a\}}) \circ d\mathbb{P}\kappa_{\varphi(a)} \circ S_{\Gamma'}(w) \quad \text{by (4.5.10)} \\ &= d\mathbb{P}\kappa_a \circ d\mathbb{P}\varphi \circ S_{\Gamma'}(w) \quad \text{since (4.5.8) commutes} \\ &= \pi_a(d\mathbb{P}\varphi(S_{\Gamma'}(w))). \end{aligned}$$

□

Consequently we have (cf. Sections 2.12 and 3.4)

4.5.13. Theorem. *Let \mathcal{V}^{Man} denote the category of virtual groupoid-invariant dynamical systems constructed above, $\pi: \mathcal{V}^{\text{Man}} \rightarrow (\text{FinGraph}/C)_{\text{et}}^{\text{op}}$ the canonical projection, $U: \text{Dynamical Systems} \rightarrow \text{Man}$ the forgetful functor, and \mathbb{P} the phase space functor extended to finite graphs (q.v. Remark 2.3.14). There exists a functor $S: \mathcal{V}^{\text{Man}} \rightarrow \text{Dynamical Systems}$ making the diagram*

$$(4.5.14) \quad \begin{array}{ccc} \mathcal{V}^{\text{Man}} & \xrightarrow{S} & \text{Dynamical Systems} \\ \pi \downarrow & & \downarrow U \\ (\text{FinGraph}/C)_{\text{et}}^{\text{op}} & \xrightarrow{\mathbb{P}} & \text{Man} \end{array}$$

commute. On objects

$$S(v, \Gamma) = (\mathbb{P}\Gamma_0, S_\Gamma(v));$$

on arrows

$$S(h, (v, \Gamma), (v', \Gamma')) = (\mathbb{P}\Gamma_0, S_\Gamma(v)) \xrightarrow{\mathbb{P}h} (\mathbb{P}\Gamma', S_{\Gamma'}(v)).$$

5. GROUPOID INVARIANCE VERSUS GROUP INVARIANCE

5.1.

Suppose the group of symmetries H acts on a colored graph $\Gamma \rightarrow C$,

$$H = \{\varphi: \Gamma \rightarrow \Gamma \mid \varphi \text{ is an isomorphism of graphs over } C\},$$

is nontrivial. Then given a phase function $\mathcal{P}: C_0 \rightarrow \text{Euc}$ we can associate to the graph Γ two kinds of invariant vector fields on the phase space $\mathbb{P}\Gamma_0$: the space virtual groupoid invariant vector fields

$\mathbb{V}(\Gamma)$ and the space $\chi(\mathbb{P}\Gamma_0)^H$ of H -invariant vector fields. In this section we show that the image of the map

$$S_\Gamma: \mathbb{V}(\Gamma) \rightarrow \chi(\mathbb{P}\Gamma_0)$$

always lands inside the space of H -invariant vector fields. In other words groupoid invariant vector fields are always group invariant. However, it is easy enough to construct examples (and we do so in this section) where the inclusion $S_\Gamma(\mathbb{V}(\Gamma)) \subset \chi(\mathbb{P}\Gamma_0)^H$ is strict. Consequently there may be many more group invariant vector fields than there are groupoid invariant vector fields, and what may be generic for group invariant vector fields need not be generic for groupoid invariant vector fields.

5.1.1. Remark. The same results with the essentially the same proofs hold in the category of manifolds (when \mathcal{P} takes values in **Man**) and in the category of finite dimensional vector spaces (when \mathcal{P} takes values in **FinVect**).

5.1.2. Proposition. *Let $\varphi: \Gamma \rightarrow \Gamma$ be an isomorphism of graphs over C . Then $\mathbb{V}(\varphi): \mathbb{V}\Gamma \rightarrow \mathbb{V}\Gamma$ is the identity map.*

Proof. Since φ is an isomorphism of graphs, it is étale. By definition (2.10.12) of $\mathbb{V}\varphi$, it is the unique linear map so that

$$(5.1.3) \quad \varpi_a \circ \mathbb{V}\varphi = \text{Ctrl}(\varphi_a)^{-1} \circ \varpi_{\varphi(a)}$$

for all vertices a of the graph Γ . But $\varphi_a: I(a) \rightarrow I(\varphi(a))$ is an arrow in the groupoid $G(\Gamma)$, hence, by definition of $\mathbb{V}\Gamma$,

$$\text{Ctrl}(\varphi_a) \circ \varpi_a = \varpi_{\varphi(a)}.$$

Consequently

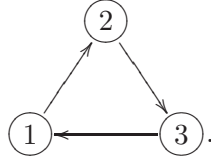
$$(5.1.4) \quad \varpi_a \circ id_{\mathbb{V}\Gamma} = \text{Ctrl}(\varphi_a)^{-1} \circ \varpi_{\varphi(a)}$$

Comparing (5.1.4) and (5.1.3) we see that $\mathbb{V}\varphi$ must be $id_{\mathbb{V}\Gamma}$. \square

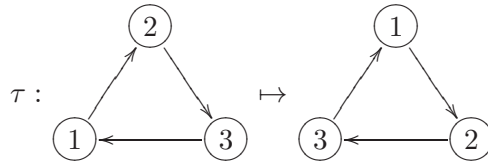
5.1.5. Theorem. *Let H be the group of automorphisms of a colored graph $\Gamma \rightarrow C$. Then the image $S_\Gamma(\mathbb{V}\Gamma)$ of virtual groupoid invariant vector fields is a subspace of H -invariant vector fields $\chi(\mathbb{P}\Gamma)^H$.*

Proof. By Theorem 2.11.2, for any virtual vector field $f \in \mathbb{V}(\Gamma)$, the vector fields $S_\Gamma(f)$ and $S_\Gamma(\mathbb{V}(h)f)$ are $\mathbb{P}h$ -related for any isomorphism $H \ni h: \Gamma \rightarrow \Gamma$. By Proposition 5.1.2, $\mathbb{V}(h)f = f$. Hence $S_\Gamma(f)$ is $\mathbb{P}h$ related to itself for any $h \in H$, i.e., is H -invariant. \square

5.1.6. Example. Let C be a graph with one vertex v and one edge: $C = \circ \curvearrowright \circ$ and let $\mathcal{P}: C_0 = \{\circ\} \rightarrow \text{Euc}$ be given by $\mathcal{P}(\circ) = \mathbb{R}$. Let Γ be a directed triangle:



Then $\mathbb{P}\Gamma = \mathbb{R}^3$. The cyclic group of order 3 generated by



is the group H of automorphisms of the graph Γ . Then $\mathbb{P}\tau: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by

$$\mathbb{P}\tau(x_1, x_2, x_3) = (x_3, x_1, x_2).$$

As before we identify the vector fields $\chi(\mathbb{R}^3)$ with smooth maps $C^\infty(\mathbb{R}^3, \mathbb{R})$. Then a vector field $F(x) = (F_1(x), F_2(x), F_3(x))$ is τ -invariant (and hence H -invariant) if and only if

$$F(\mathbb{P}\tau(x)) = \mathbb{P}\tau(F(x)).$$

Hence

$$\begin{aligned} F_1(x_2, x_3, x_1) &= F_2(x_1, x_2, x_3) \\ F_2(x_2, x_3, x_1) &= F_3(x_1, x_2, x_3). \end{aligned}$$

Therefore the space $\chi(\mathbb{P}\Gamma_0)^H$ of H -invariant vector fields is isomorphic to $C^\infty(\mathbb{R}^3, \mathbb{R})$. The isomorphism is given by

$$\begin{aligned} C^\infty(\mathbb{R}^3, \mathbb{R}) &\rightarrow C^\infty(\mathbb{R}^3, \mathbb{R})^H = \chi(\mathbb{P}\Gamma)^H \\ h(x) &\mapsto (h(x), h(\mathbb{P}\tau x), h((\mathbb{P}\tau)^2 x)). \end{aligned}$$

On the other hand, all the input trees of Γ are of the form



and are all isomorphic to each other. We conclude two things: for any vertex a of Γ , $\text{Ctrl}(I(a)) \simeq C^\infty(\mathbb{R}, \mathbb{R})$ and the groupoid $G(\Gamma)$ is equivalent to the trivial category with one object and one arrow. Hence by Theorem 2.10.17 the space $\mathbb{V}(\Gamma)$ can be identified with the vector space $C^\infty(\mathbb{R}, \mathbb{R})$. Tracing through the identifications we see that $S_\Gamma: \mathbb{V}\Gamma \rightarrow \chi(\mathbb{P}\Gamma)$ is given by the injective map

$$\begin{aligned} S_\Gamma: \mathbb{V}\Gamma \simeq C^\infty(\mathbb{R}, \mathbb{R}) &\rightarrow C^\infty(\mathbb{R}^3, \mathbb{R}) \simeq \chi(\mathbb{P}\Gamma)^H \\ f(u) &\mapsto (f(x_3), f(x_1), f(x_2)). \end{aligned}$$

Clearly the space of groupoid-invariant vector fields $\mathbb{V}\Gamma$ is much smaller than the space of group-invariant vector fields $\chi(\mathbb{P}\Gamma)^H$.

6. FUNCTORIALITY OF LIMITS

6.1.

The goal of this section is to set up the category theoretic framework alluded to in 2.10.14 and prove 2.10.15 and 2.10.17.

6.1.1. Notation. The symbol \mathbf{Cat} denotes the collection of all small categories.

6.1.2. It is not too wrong to think of \mathbf{Cat} as a category whose objects are small categories and morphisms are functors. But it is also not completely accurate since there are also natural transformations between functors.

Fix a category \mathbf{D} . Assume that for any functor $f: \mathbf{C} \rightarrow \mathbf{D}$, with \mathbf{C} small, the limit $\lim(f: \mathbf{C} \rightarrow \mathbf{D})$ exists in \mathbf{D} , that is, assume that \mathbf{D} is a *complete* category (q.v. Definition A.2.3). The category \mathbf{D} itself need not be small. For our applications we want \mathbf{D} to be the category \mathbf{Vect} of not necessarily finite dimensional vector spaces, but nothing that we do in this section requires \mathbf{D} to be \mathbf{Vect} .

Consider the new category \mathbf{Cat}/\mathbf{D} with objects

$$(\mathbf{Cat}/\mathbf{D})_0 = \{X \xrightarrow{a} \mathbf{D} \mid X \text{ is a small category, } a \text{ is a functor}\}.$$

A morphism in \mathbf{Cat}/\mathbf{D} from $Y \xrightarrow{b} \mathbf{D}$ to $X \xrightarrow{a} \mathbf{D}$ is a pair (f, α) where $f: Y \rightarrow X$ is a functor and $\alpha: af \Rightarrow b$ a natural transformation. We picture morphisms as 2-commutative triangles:

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 & \searrow \alpha \swarrow & \\
 & D &
 \end{array}
 \begin{array}{c}
 b \swarrow \\
 a \searrow
 \end{array}$$

The composition $*$ of morphisms in \mathbf{D} is defined by

$$(f, \alpha) * (g, \beta) = (fg, \beta \circ_V (\alpha \circ_H g))$$

for any pair $(g, \beta): (Z \xrightarrow{c} D) \rightarrow (Y \xrightarrow{b} D)$, $(f, \alpha): (Y \xrightarrow{b} D) \rightarrow (X \xrightarrow{a} D)$ of composable morphisms of \mathbf{Cat}/\mathbf{D} . Here \circ_V and \circ_H denote the vertical and horizontal compositions (see A.1.32). We picture the composition $*$ as pasting of two 2-commuting triangles:

$$\left(\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \alpha \swarrow & \\ & D & \end{array} \begin{array}{c} b \swarrow \\ a \searrow \end{array}, \begin{array}{ccc} Z & \xrightarrow{g} & Y \\ & \searrow \beta \swarrow & \\ & D & \end{array} \begin{array}{c} c \swarrow \\ b \searrow \end{array} \right) \xrightarrow{*} \begin{array}{ccc} Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \\ & \searrow \beta \swarrow & \downarrow b & \searrow \alpha \swarrow & \\ & D & & D & \end{array} = \begin{array}{ccc} Z & \xrightarrow{fg} & X \\ & \searrow \swarrow & \\ & D & \end{array} \begin{array}{c} c \swarrow \\ b \searrow \end{array} .$$

One checks that $*$ is associative and that \mathbf{Cat}/\mathbf{D} is a category.

6.1.3. Proposition. *We use the notation above. The assignment*

$$L: (\mathbf{Cat}/\mathbf{D})_0 \rightarrow \mathbf{D}_0, \quad L(X \xrightarrow{a} D) := \lim(X \xrightarrow{a} D)$$

extends to a contravariant functor

$$L: (\mathbf{Cat}/\mathbf{D})^{\text{op}} \rightarrow \mathbf{D}.$$

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 & \searrow \alpha \swarrow & \\
 & D &
 \end{array}
 \begin{array}{c}
 b \swarrow \\
 a \searrow
 \end{array}$$

Proof. Consider a morphism in \mathbf{Cat}/\mathbf{D} . Let $p_x: \lim(X \xrightarrow{a} D) \rightarrow a(x)$, $x \in X_0$ and

$q_y: \lim(Y \xrightarrow{b} D) \rightarrow b(y)$, $y \in Y_0$ denote the canonical projections. For any arrow $y \xrightarrow{\gamma} y'$ in Y we have a commuting diagram

$$\begin{array}{ccccc}
 \lim(X \xrightarrow{a} D) & & & & \\
 \searrow p_{f(y)} & \searrow p_{f(y')} & & & \\
 & af(y) \xrightarrow{af(\gamma)} af(y') & & & \\
 \alpha_y \downarrow & \downarrow \alpha_{y'} & & & \\
 & b(y) \xrightarrow{b(\gamma)} b(y') & & & \\
 q_y \nearrow & \nearrow q_{y'} & & & \\
 \lim(Y \xrightarrow{b} D) & & & &
 \end{array}$$

By the universal property of $\{q_y: \lim(Y \xrightarrow{b} D) \rightarrow b(y)\}_{y \in Y_0}$ we have a unique map $L(f, \alpha): \lim(X \xrightarrow{a} D) \rightarrow \lim(Y \xrightarrow{b} D)$ making the diagram

$$\begin{array}{ccc}
 \lim(X \xrightarrow{a} D) & & \\
 \downarrow p_{f(y)} & \searrow p_{f(y')} & \\
 & af(y) \xrightarrow{af(\gamma)} af(y') & \\
 \downarrow \alpha_y & & \downarrow \alpha_{y'} \\
 & b(y) \xrightarrow{b(\gamma)} b(y') & \\
 \uparrow q_y & & \uparrow q_{y'} \\
 \lim(Y \xrightarrow{b} D) & &
 \end{array}$$

$\exists! L(f, \alpha)$

commute. The fact that L preserves the composition of morphisms

$$\begin{array}{ccccc}
 Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \\
 & \searrow \beta & \downarrow b & \swarrow \alpha & \\
 & & D & &
 \end{array}$$

c a

follows from the commutativity of the diagram below

$$\begin{array}{ccccc}
 \lim(X \xrightarrow{a} D) & & & & \\
 \downarrow L(f, \alpha) & \searrow p_{fg(z)} & \searrow p_{fg(z')} & & \\
 & afg(z) \xrightarrow{afg(\gamma)} afg(z') & & & \\
 \downarrow q_{g(z)} & & \downarrow q_{g(z')} & & \\
 \lim(Y \xrightarrow{b} D) & \xrightarrow{q_g(z)} bg(z) \xrightarrow{bg(\gamma)} bg(z') & & & \\
 \downarrow L((g, \beta) * (f, \alpha)) & & \downarrow c(z) \xrightarrow{c(\gamma)} c(z') & & \\
 \downarrow L(g, \beta) & & \downarrow r_z & \searrow r_{z'} & \\
 \lim(Z \xrightarrow{c} D) & & & &
 \end{array}$$

for any arrow $z \xrightarrow{\gamma} z'$ in Z . □

6.1.4. Remark. It follows from the proof above that if $a, a': X \rightarrow D$ is a pair of functors and $\alpha: a \Rightarrow a'$ is a natural *isomorphism*, then $L(id, \alpha): L(X \xrightarrow{a} D) \rightarrow L(X \xrightarrow{a'} D)$ is an isomorphism in

D. Since any 2-commutative triangle

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow b & \swarrow a \\ & D & \end{array} \quad \begin{array}{c} \nearrow \alpha \\ \nwarrow \end{array}$$

can be factored as

$$\begin{array}{ccccc} Y & \xrightarrow{id_Y} & Y & \xrightarrow{f} & X \\ & \searrow b & \downarrow af & \swarrow a & \\ & D & & & \end{array} \quad \begin{array}{c} \nearrow \alpha \\ \nwarrow \end{array},$$

if α is a natural isomorphism then $L(f, \alpha)$ is an isomorphism if and only if $L(f, id)$ is an isomorphism.

We are now ready to state and prove the main result of the section.

6.1.5. Theorem. *Let X, Y be small categories and D a category that has all small limits. Suppose $Y \xrightarrow{F} X$ is part of an equivalence of categories and $\alpha: aF \Rightarrow b$ a natural isomorphism. Then*

$$L(F, \alpha): \lim(X \xrightarrow{a} D) \rightarrow \lim(Y \xrightarrow{b} D)$$

is an isomorphism in D .

Note that by Remark 6.1.4 it is no loss of generality to assume that α is the identity natural isomorphism, that is, that $af = b$.

There are several ways to write down the proof of Theorem 6.1.5. We present the proof that uses nothing more than the universal property of limits. The reader may wish to write down a shorter proof that uses adjoint functors.

Proof of Theorem 6.1.5. Without loss of generality we assume that α is the identity natural isomorphism so that $b = af$. Let $\{p_x: L \rightarrow a(x)\}_{x \in X}$ denote a limit of $a: X \rightarrow D$. Then by definition for any arrow $y \xrightarrow{f} y'$ of Y the diagram

$$\begin{array}{ccc} & L & \\ p_{F(y)} \swarrow & & \searrow p_{F(y')} \\ aF(y) & \xrightarrow{aF(f)} & aF(y') \end{array}$$

commutes in D . We argue that $\{p_{F(y)}: L \rightarrow aF(y)\}_{y \in Y_0}$ is the terminal object in $\text{Cone}(aF)$, that is, a limit of $aF: Y \rightarrow D$. For then the induced map $L(F, id): L \rightarrow L$ is the identity map and the result follows.

Suppose $\{\mu_a: d \rightarrow aF(y)\}_{y \in Y_0}$ is a cone on aF . We need to produce a morphism of cones $\mu: (d, \{\mu_y\}) \rightarrow (L, \{p_{F(y)}\})$. Since $F: Y \rightarrow X$ is part of an equivalence of categories there is a functor

$$H: X \rightarrow Y$$

and natural isomorphism

$$\tau: FH \Rightarrow id_X, \quad \nu: HF \Rightarrow id_Y.$$

For any $x \in X_0$ the arrow $\tau_x: FH(x) \rightarrow x$ is an isomorphism in X . Hence $a(\tau_x): aFH(x) \rightarrow a(x)$ is an isomorphism in D . Consider

$$\zeta_x: d \rightarrow a(x), \quad \zeta_x := a(\tau_x) \circ \mu_{H(x)}.$$

For any arrow $x \xrightarrow{g} x'$ in \mathbf{X} the square

$$\begin{array}{ccc} FH(x) & \xrightarrow{FH(g)} & FH(x') \\ \tau_x \downarrow & & \downarrow \tau_{x'} \\ x & \xrightarrow{g} & x' \end{array}$$

commutes in \mathbf{X} . Hence

$$(6.1.6) \quad \begin{array}{ccc} aFH(x) & \xrightarrow{aFH(g)} & aFH(x') \\ a(\tau_x) \downarrow & & \downarrow a(\tau_{x'}) \\ a(x) & \xrightarrow{a(g)} & a(x') \end{array}$$

commutes in \mathbf{D} . By assumption

$$(6.1.7) \quad \begin{array}{ccc} & d & \\ \mu_{H(x)} \swarrow & & \searrow \mu_{H(x')} \\ aFH(x) & \xrightarrow{aFH(g)} & aFH(x') \end{array}$$

commutes in \mathbf{D} . Equations (6.1.6) and (6.1.7) together with the definition of ζ_x imply that

$$\begin{array}{ccc} & d & \\ \zeta_x \swarrow & & \searrow \zeta_{x'} \\ a(x) & \xrightarrow{a(g)} & a(x') \end{array}$$

commutes in \mathbf{D} , that is, $(d, \{\zeta_x\})$ is a cone over a . Since $(L, \{p_x\})$ is terminal in $\text{Cone}(a)$, there is a unique map of cones $\zeta: (d, \{\zeta_x\}) \rightarrow (L, \{p_x\})$. Hence

$$(6.1.8) \quad \begin{array}{ccc} d & \xrightarrow{\zeta} & L \\ \zeta_{F(y)} \swarrow & & \searrow p_{F(y)} \\ & aF(y) & \end{array}$$

commutes for any object y of \mathbf{Y} . However,

$$\zeta_{F(y)} = a(\tau_{F(y)}) \circ \mu_{HF(y)},$$

which is not μ_y . Indeed, since $HF(y) \xrightarrow{\nu_y} y$ is an arrow in \mathbf{Y} (in fact an isomorphism), the diagram

$$\begin{array}{ccc} & d & \\ \mu_{HF(y)} \swarrow & & \searrow \mu_y \\ aFHF(y) & \xrightarrow{aF(\nu_y)} & aF(y) \end{array}$$

commutes. Hence

$$(6.1.9) \quad \zeta_{F(y)} = a(\tau_{F(y)}) \circ aF(\nu_y^{-1}) \circ \mu_y$$

for any y in \mathbf{Y} . We therefore can rewrite (6.1.8) as the commuting diagram

$$(6.1.10) \quad \begin{array}{ccc} d & \xrightarrow{\zeta} & L \\ \mu_y \downarrow & & \downarrow p_{F(y)} \\ aF(y) & \xleftarrow{\kappa_y} & aF(y) \end{array},$$

where

$$\kappa_y := (a(\tau_{F(y)}) \circ aF(\nu_y^{-1}))^{-1},$$

for any object y of \mathbf{Y} . Note that since τ and ν are natural isomorphisms, the collection $\{\kappa_y\}$ is a natural isomorphism from aF to aF . Consequently

$$(L, \{\kappa_y \circ p_{F(y)}\}_{y \in \mathbf{Y}_0})$$

is a cone on aF . Moreover the preceding argument shows that for any cone $(d, \{\mu_y\})$ on aF there is a morphism of cones ζ from $(d, \{\mu_y\})$ to $(L, \{\kappa_y \circ p_{F(y)}\})$. We leave it to the reader to check that ζ is, in fact, unique.

We conclude that $(L, \{\kappa_y \circ p_{F(y)} : L \rightarrow aF(y)\}_{y \in \mathbf{Y}_0})$ is a limit of $aF : \mathbf{Y} \rightarrow \mathbf{D}$, that is, a terminal object in $\mathbf{Cone}(aF)$. The natural isomorphism $\kappa : aF \Rightarrow aF$ induces an isomorphism of categories $\mathbf{Cone}(aF) \rightarrow \mathbf{Cone}(aF)$. It follows that since $(L, \{\kappa_y \circ p_{F(y)} : L \rightarrow aF(y)\}_{y \in \mathbf{Y}_0})$ is a terminal object in $\mathbf{Cone}(aF)$, so is $(L, \{p_{F(y)} : L \rightarrow aF(y)\}_{y \in \mathbf{Y}_0})$. \square

Proof of Theorem 2.10.15. Since $G(\varphi) : G(\Gamma) \rightarrow G(\Gamma')$ is always fully faithful by construction, the assumption that $G(\varphi)$ is essentially surjective implies that $G(\varphi)$ is an equivalence of categories. By (2.10.11) the diagram

$$\begin{array}{ccc} G(\Gamma) & \xrightarrow{G(\varphi)} & G(\Gamma') \\ & \searrow \phi & \swarrow \\ \text{Ctrl}|_{G(\Gamma)} & & \text{Ctrl}|_{G(\Gamma')} \\ & \searrow & \swarrow \\ & \text{Vect} & \end{array}$$

2-commutes (q.v. Remark 2.10.14). By Theorem 6.1.5 $\mathbb{V}(\varphi)$ is an isomorphism. \square

Proof of Theorem 2.10.17. Let $\varphi : \bigsqcup_{i=1}^k \text{Aut}(I(a_i)) \hookrightarrow G(\Gamma)$ denote the canonical inclusion of the skeleton of the groupoid $G(\Gamma)$ into the groupoid. By definition of the skeleton, φ is an equivalence of categories. Clearly

$$\begin{array}{ccc} \bigsqcup_{i=1}^k \text{Aut}(I(a_i)) & \xrightarrow{G(\varphi)} & G(\Gamma) \\ & \searrow & \swarrow \\ \text{Ctrl}|_{\bigsqcup_{i=1}^k \text{Aut}(I(a_i))} & & \text{Ctrl}|_{G(\Gamma)} \\ & \searrow & \swarrow \\ & \text{Vect} & \end{array}$$

strictly commutes. Hence by Theorem 6.1.5

$$\mathbb{V}(\varphi) : \mathbb{V}(\Gamma) \rightarrow \lim(\text{Ctrl} : \bigsqcup_{i=1}^k \text{Aut}(I(a_i)) \rightarrow \text{Vect})$$

is an isomorphism. But

$$\lim(\text{Ctrl} : \bigsqcup_{i=1}^k \text{Aut}(I(a_i)) \rightarrow \text{Vect}) = \prod_{i=1}^k \text{Ctrl}(I(a_i))^{\text{Aut}(I(a_i))}$$

by A.2.13, and the result follows. \square

APPENDIX A. ELEMENTS OF CATEGORY THEORY

A.1. Basic notions.

We start by recalling the basic definitions of category theory, mostly to fix our notation. This appendix may be useful to the reader with some background in category theory; the reader with little to no experience in category theory may wish to consult a textbook such as [63].

A.1.1. Definition (Category). A **category** \mathbf{A} consists

- (1) A collection⁷ A_0 of *objects*;
- (2) For any two objects $a, b \in A_0$, a set $\text{Hom}_A(a, b)$ of *morphisms* (or *arrows*);
- (3) For any three objects $a, b, c \in A_0$, and any two arrows $f \in \text{Hom}_A(a, b)$ and $g \in \text{Hom}_A(b, c)$, a *composite* $g \circ f \in \text{Hom}_A(a, c)$, i.e., for all triples of objects $a, b, c \in A_0$ there is a *composition map*

$$\circ: \text{Hom}_A(b, c) \times \text{Hom}_A(a, b) \rightarrow \text{Hom}_A(a, c),$$

$$\text{Hom}_A(b, c) \times \text{Hom}_A(a, b) \ni (g, f) \mapsto g \circ f \in \text{Hom}_A(a, c).$$

This composition operation is *associative* and has *units*, that is,

- i. for any triple of morphisms $f \in \text{Hom}_A(a, b)$, $g \in \text{Hom}_A(b, c)$ and $h \in \text{Hom}_A(c, d)$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

- ii. for any object $a \in A_0$, there exists a morphism $1_a \in \text{Hom}_A(a, a)$, called the *identity*, which is such that for any $f \in \text{Hom}_A(a, b)$ we have

$$f = f \circ 1_a = 1_b \circ f.$$

We denote the collection of all morphisms of a category A by A_1 :

$$A_1 = \bigsqcup_{a, b \in A_0} \text{Hom}_A(a, b).$$

A.1.2. Remark. The symbol “ \circ ” is customarily suppressed in writing out compositions of two morphisms. Thus

$$gf \equiv g \circ f.$$

A.1.3. Remark. Given a category A its underlying graph has the collections A_0 as nodes, A_1 as edges, and the source and target maps $s, t: A_1 \rightarrow A_0$ are defined so that for any $f \in \text{Hom}_A(a, b)$ we have

$$s(f) = a, \quad t(f) = b.$$

In other words, if we forget the composition in a category we are left with a (directed) graph.

A.1.4. Example (Category **Set** of sets). The collection **Set** of all sets forms a category. The objects of **Set** are sets, the arrows of **Set** are ordinary maps and the composition of arrows is the composition of maps.

A.1.5. Example (Category **Vect** of vector spaces). The collection **Vect** of all real vector spaces (not necessarily finite dimensional) forms a category. Its objects are vector spaces and its morphisms are linear maps. The composition of morphisms is the ordinary composition of linear maps.

A.1.6. Example (Category **Graph** of directed graphs). The collection **Graph** of all directed graphs (Definition 2.2.1) forms a category. Its objects are graphs and its morphisms are maps of graphs.

A.1.7. Example (Category **Man** of manifolds). The collection of all finite dimensional Hausdorff paracompact manifolds and smooth maps forms the category **Man** of manifolds.

A.1.8. Example (Sets as discrete categories). Any set X can be thought of as a category X with the collection of objects $X_0 = X$ and the collection of arrows $X_1 = \{1_x\}_{x \in X}$. The only pairs of arrows that can be composed are of the form $(1_x, 1_x)$ and we define their composite to be 1_x . One refers to such categories as *discrete* categories.

A.1.9. Definition. A *subcategory* A of a category B is a collection of some objects A_0 and some arrows A_1 of B such that:

⁷A collection may be too big to be a set. While for many constructions in category theory the *size* of A_0 and A_1 is important, it will play only a limited role in this paper. Thus we will mostly ignore the usual set-theoretic issues.

- For each object $a \in A_0$, the identity 1_a is in A_1 ;
- For each arrow $f \in A_1$ its source and target $s(f), t(f)$ are in A_0 ;
- for each pair $(f, g) \in A_0 \times A_0$ of composable arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ the composite $g \circ f$ is in A_1 as well.

A.1.10. **Remark.** Naturally a subcategory is a category in its own right.

A.1.11. **Example.** The collection **FinSet** of all finite sets and all maps between them is a subcategory of **Set** hence a category.

The collection **FinGraph** of finite directed graphs (that is graphs whose collections of nodes and edges are finite sets) and maps of graphs between them is a subcategory of **Graph**.

The collection **FinVect** of real finite dimensional vector spaces and linear maps is a subcategory of **Vect**.

A.1.12. **Example** (The category **Euc** of Euclidean spaces). The collection of all finite dimensional real vector spaces is a collection of objects for two different categories:

- (1) It is the collection of objects of **FinVect**.
- (2) Since every finite dimensional vector space is canonically a Hausdorff second countable manifold, we also have the category **Euc** (“Euclidean spaces”) with the same objects as **FinVect** but with morphisms defined to be all smooth maps.

The category **FinVect** can be thought of as a subcategory of the category **Euc** of Euclidean spaces. The category of Euclidean spaces is naturally a subcategory of the category **Man** of manifolds.

A.1.13. **Example** (Groups are categories). In contrast with sets, which are categories with many objects and almost no morphisms, a group G can be viewed as a category **G** with one object as follows: the set of objects of **G** is a set with one element: $G_0 = \{*\}$. The set of arrows G_1 is the set of elements of the group G : $G_1 = G$. The composition of arrows in **G** is multiplication in the group G .

A.1.14. **Example** (Disjoint union of groups as a category). Let $\{G_\alpha\}_{\alpha \in A}$ be a family of groups index by a set A . The disjoint union $\bigsqcup_{\alpha \in A} G_\alpha$ is not a group, since we cannot multiply elements of two different groups, but it can be thought of as a category **A**. Here are the details.

The set of objects of the category **A** is the set A : $A_0 = A$. The set of morphisms A_1 is the disjoint union $\bigsqcup_{\alpha \in A} G_\alpha$ with $\text{Hom}_A(\alpha, \alpha') = \emptyset$ if $\alpha \neq \alpha'$ and $\text{Hom}_A(\alpha, \alpha) = G_\alpha$. The composition in $\text{Hom}_A(\alpha, \alpha)$ is the multiplication in G_α .

Elsewhere in the paper we abuse the notation and write $\bigsqcup_{\alpha \in A} G_\alpha$ when we mean the category **A**.

A.1.15. **Definition** (isomorphism). An arrow $f \in \text{Hom}_A(a, b)$ in a category **A** is an *isomorphism* if there is an arrow $g \in \text{Hom}_A(b, a)$ with $g \circ f = 1_a$ and $f \circ g = 1_b$. We think of f and g as inverses of each other and may write $g = f^{-1}$. Clearly $g = f^{-1}$ is also an isomorphism.

Two objects $a, b \in A_0$ are *isomorphic* if there is an isomorphism $f \in \text{Hom}_A(a, b)$. We will also say that a is isomorphic to b .

A.1.16. **Definition** (Groupoid). A *groupoid* is a category in which every arrow is an isomorphism.⁸

A.1.17. **Example.** Sets (thought of as discrete categories), groups (thought of as categories with just one object), and disjoint unions of groups A.1.14 are all groupoids.

A.1.18. **Definition.** A category **X** is *small* if its collections X_0 of objects and X_1 of morphisms are both sets.

⁸Readers uneasy about the issues of size may further restrict the definition of a groupoid by requiring that its collection of objects is a set.

A.1.19. **Example.** Any group thought of as a category is a small category. Any set thought of as a discrete category is a small category. However, the categories **Set**, **FinSet**, **Vect**, **Euc**, and **Man** are not small.

A.1.20. **Definition** (Opposite category). Given a category **A**, the *opposite category* \mathbf{A}^{op} has the same objects as **A** and the arrows are reversed. That is $(\mathbf{A}^{\text{op}})_0 = \mathbf{A}_0$ and for any two objects $a, b \in \mathbf{A}_0 = (\mathbf{A}^{\text{op}})_0$ the set of morphisms is defined by

$$\text{Hom}_{\mathbf{A}^{\text{op}}}(a, b) = \text{Hom}_{\mathbf{A}}(b, a),$$

The composition

$$\circ^{\text{op}}: \text{Hom}_{\mathbf{A}^{\text{op}}}(b, c) \times \text{Hom}_{\mathbf{A}^{\text{op}}}(a, b) \rightarrow \text{Hom}_{\mathbf{A}^{\text{op}}}(a, c)$$

in \mathbf{A}^{op} is defined by

$$\text{Hom}_{\mathbf{A}^{\text{op}}}(b, c) \times \text{Hom}_{\mathbf{A}^{\text{op}}}(a, b) (= \text{Hom}_{\mathbf{A}}(c, b) \times \text{Hom}_{\mathbf{A}}(b, a)) \ni (g, f) \mapsto g \circ^{\text{op}} f := f \circ g.$$

A.1.21. **Definition** (Slice category). Given a subcategory **A** of a category **B** and an object $b \in \mathbf{B}_0$ we can form a new category, \mathbf{A}/b . The objects of \mathbf{A}/b are arrows of **B** of the form $x \xrightarrow{\alpha} b$ with $x \in \mathbf{A}_0$. A morphism from $x \xrightarrow{\alpha} b$ to $y \xrightarrow{\beta} b$ is defined to be a commuting triangle

$$\begin{array}{ccc} x & \xrightarrow{h} & y \\ & \searrow \alpha & \swarrow \beta \\ & b & \end{array}$$

in **B** (cf. Mac Lane [52, p. 45]). The category \mathbf{A}/b is called the *slice category* and the *comma category*. In this paper we will often abbreviate the commuting triangle above as $h: x \rightarrow y$ with the rest of the triangle understood.

A.1.22. **Remark.** The two slice categories that are important for us in this paper are \mathbf{FinSet}/C_0 where C_0 is a set, which is not necessarily finite, and $\mathbf{FinGraph}/C$ where C is a directed graph, again, not necessarily finite.

A.1.23. **Definition** (Functor). A (covariant) *functor* $F: \mathbf{A} \rightarrow \mathbf{B}$ from a category **A** to a category **B** is a map on the objects and arrows of **A** such that every object $a \in \mathbf{A}_0$ is assigned an object $Fa \in \mathbf{B}_0$, every arrow $f \in \text{Hom}_{\mathbf{A}}(a, b)$ is assigned an arrow $Ff \in \text{Hom}_{\mathbf{B}}(Fa, Fb)$, and such that composition and identities are preserved, namely

$$F(f \circ g) = Ff \circ Fg, \quad F1_a = 1_{Fa}.$$

A *contravariant* functor from **A** to **B** is a covariant functor $G: \mathbf{A}^{\text{op}} \rightarrow \mathbf{B}$. This amounts to:

$$G(f \circ g) = G(g) \circ G(f)$$

for all composable pairs of arrows f, g of **A**.

A.1.24. **Example.** Given a category **A** there is the *identity functor* $1_{\mathbf{A}}: \mathbf{A} \rightarrow \mathbf{A}$ which is the identity on objects and arrows.

A.1.25. **Example.** If **A** is a subcategory of **B** then the natural inclusion $i: \mathbf{A} \rightarrow \mathbf{B}$ is a functor.

A.1.26. **Remark.** Since functors are maps, functors can be composed.

A.1.27. **Remark.** A functor $F: \mathbf{X} \rightarrow \mathbf{A}$ from a *discrete* category **X** to a category **A** is a map F from the set X of objects of **X** to the collection of objects of **A**: $F: X \rightarrow \mathbf{A}_0$.

A.1.28. **Definition.** A functor $F: \mathbf{A} \rightarrow \mathbf{B}$ is

- (1) *full* if $F: \text{Hom}_{\mathbf{A}}(a, a') \rightarrow \text{Hom}_{\mathbf{B}}(Fa, Fa')$ is surjective for all pairs of objects $a, a' \in \mathbf{A}_0$;
- (2) *faithful* if $F: \text{Hom}_{\mathbf{A}}(a, a') \rightarrow \text{Hom}_{\mathbf{B}}(Fa, Fa')$ is injective for all pairs of objects $a, a' \in \mathbf{A}_0$

- (3) *fully faithful* if $F: \text{Hom}_A(a, a') \rightarrow \text{Hom}_B(Fa, Fa')$ is a bijection for all pairs of objects $a, a' \in A_0$;
- (4) *essentially surjective* if for any object $b \in B_0$ there is an object $a \in A_0$ and an isomorphism $f \in \text{Hom}_B(F(a), b)$. That is, for any object b of B there is an object a of A so that b and $F(a)$ are isomorphic.

A.1.29. **Example.** The inclusion $i: \text{Euc} \hookrightarrow \text{Man}$ is fully faithful. It is not essentially surjective since not every manifold is diffeomorphic to a vector space.

The “inclusion” $\text{FinVect} \hookrightarrow \text{Euc}$ is faithful but not full since not every smooth map is linear. It is essentially surjective since FinVect and Euc have the “same” objects.

A.1.30. **Definition.** A subcategory A of a category B is *full* if the inclusion functor $i: A \hookrightarrow B$ is full (it is faithful by definition of what it means to be a subcategory). That is, for any two objects a, a' of A ,

$$\text{Hom}_A(a, a') = \text{Hom}_B(a, a').$$

A.1.31. **Definition** (Natural Transformation). Let $F, G: A \rightarrow B$ be a pair of functors. A *natural transformation* $\tau: F \Rightarrow G$ is a family of $\{\tau_a: Fa \rightarrow Ga\}_{a \in A_0}$ of morphisms in B , one for each object a of A , such that, for any $f \in \text{Hom}_A(a, a')$, the following diagram commutes:

$$\begin{array}{ccc} Fa & \xrightarrow{Ff} & Fb \\ \tau_a \downarrow & & \downarrow \tau_b \\ Ga & \xrightarrow{Gg} & Gb \end{array}$$

If each τ_a is an isomorphism, we say that τ is a *natural isomorphism* (an older term is *natural equivalence*).

A.1.32. **Definition.** Natural transformations and functors can be composed in several distinct ways. We will need two.

If $f, g, h: Y \rightarrow X$ are functors and $\alpha: f \Rightarrow g, \beta: g \Rightarrow h$ are natural transformations, then their *vertical composition* $\beta \circ_V \alpha: f \Rightarrow h$ is defined by

$$(\beta \circ_V \alpha)_y := \beta_y \circ \alpha_y$$

for all $y \in Y_0$. The composition \circ on the right is the composition of arrows in X .

If $k: Z \rightarrow Y, f, g: Y \rightarrow X$ are functors and $\alpha: f \Rightarrow g$ a natural transformation, we define the *horizontal composition* $\alpha \circ_H k: fk \Rightarrow gk$ by

$$(\alpha \circ_H k)_z := \alpha_{k(z)}$$

for all $z \in Z_0$.

A.1.33. **Definition** (Equivalence of categories). An *equivalence of categories* consists of a pair of functors

$$F: A \rightarrow B, \quad E: B \rightarrow A$$

and a pair of natural isomorphisms

$$\alpha: 1_A \Rightarrow E \circ F \quad \beta: 1_B \Rightarrow F \circ E.$$

In this situation the functor F is called *the pseudo-inverse* or the *homotopy inverse* of E . The categories A and B are then said to be *equivalent*.

A.1.34. **Proposition.** A functor $F: A \rightarrow B$ is (part of) an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*.

Proof. See [63, Proposition 7.25]

□

A.1.35. **Example** (Any groupoid is equivalent to a disjoint union of groups).⁹ Suppose a category \mathbf{B} is a groupoid. Then either two objects b, b' of \mathbf{B} are isomorphic or $\text{Hom}_{\mathbf{B}}(b, b')$ is empty. Thus “being isomorphic” is an equivalence relation on the collection of objects \mathbf{B}_0 of \mathbf{B} . Pick one representative for each equivalence class of objects of \mathbf{B} and call this collection A . For every $a \in A$, $\text{Hom}_{\mathbf{B}}(a, a)$ is a group; call it G_a . We have a natural inclusion functor $i: \bigsqcup_{a \in A} G_a \rightarrow \mathbf{B}$. The functor i is fully faithful by construction. It is essentially surjective since any object $b \in \mathbf{B}_0$ is isomorphic to some object $a \in (\bigsqcup_{a \in A} G_a)_0 = A$ by construction of A . By Proposition A.1.34 above, i is an equivalence of categories.

We will refer to the disjoint union $\bigsqcup_{a \in A} G_a$ as a *skeleton* of the groupoid \mathbf{B} .

A.2. Limits.

A.2.1. **Definition** (Limit). A *limit* $\lim F \equiv \lim(F: \mathbf{A} \rightarrow \mathbf{B})$ of a functor $F: \mathbf{A} \rightarrow \mathbf{B}$ (if it exists!) is an object $\lim F$ of \mathbf{B} together with a collection of arrows $\{p_a: \lim F \rightarrow F(a)\}_{a \in \mathbf{A}_0}$ (called projections) so that

- (1) for any arrow $a \xrightarrow{f} a'$ in \mathbf{A} the diagram

$$\begin{array}{ccc} & \lim F & \\ p_a \swarrow & & \searrow p_{a'} \\ F(a) & \xrightarrow{F(f)} & F(a') \end{array}$$

commutes;

- (2) Given an object b of \mathbf{B} and a family of arrows $\{\xi_a: b \rightarrow F(a)\}_{a \in \mathbf{A}_0}$ so that

$$\begin{array}{ccc} & b & \\ \xi_a \swarrow & & \searrow \xi_{a'} \\ F(a) & \xrightarrow{F(f)} & F(a') \end{array}$$

commutes for any arrow $f \in \text{Hom}_{\mathbf{A}}(a, a')$, there is a unique arrow $\xi: b \rightarrow \lim F$ making the diagram

$$\begin{array}{ccc} \lim F & \xleftarrow{\quad \xi \quad} & b \\ p_a \downarrow & \swarrow p_{a'} \quad \searrow \xi_a & \downarrow \xi_{a'} \\ F(a) & \xrightarrow{F(f)} & F(a') \end{array}$$

commute.

A.2.2. **Remark** (“Uniqueness” of limits). It is not hard to show using property (2) of the definition of a limit, that while limits are not unique, any two limits of the same functor F are isomorphic by way of a unique isomorphism. One says that limits are unique up to a unique isomorphism.

We will occasionally gloss over this fine point and talk about “the limit” of a functor F .

A.2.3. **Definition.** A category \mathbf{D} is *complete* if for any *small* category \mathbf{X} and any functor $f: \mathbf{X} \rightarrow \mathbf{D}$ the limit $\lim(f: \mathbf{X} \rightarrow \mathbf{D})$ exists in \mathbf{D} .

It is known that the categories \mathbf{Set} of sets and the category \mathbf{Vect} of vector spaces are complete [52], [63]. The category of manifolds \mathbf{Man} is not complete.

A categorical product is a special kind of a limit. We will first define categorical products, then give examples and then explain why they (products) are limits.

⁹This is a special case of the fact that any category is equivalent to its skeleton [52, p. 93]

A.2.4. Definition (Categorical product). A *categorical product* of a family $\{F_x\}_{x \in X}$ of objects of a category \mathbf{B} indexed by the elements of a set X is an object $\prod_{x \in X} F_x$ of \mathbf{B} together with a set of arrows $\{p_x: \prod_{y \in X} F_y \rightarrow F_x\}_{x \in X}$ of \mathbf{B} enjoying the following universal property: for any object b of \mathbf{B} and any set of maps $\{\xi_x: b \rightarrow F_x\}$ there is a unique map $\xi: b \rightarrow \prod_{x \in X} F_x$ making the diagram

$$\begin{array}{ccc} \prod_{y \in X} F_y & \xleftarrow{\exists! \xi} & b \\ p_x \downarrow & \swarrow \xi_x & \\ F_x & & \end{array}$$

commute.

A.2.5. Remark. It is easy to show using the universal property of categorical products that categorical products (if they exist) are unique up to a unique isomorphisms. Again, we will gloss over non-uniqueness of products and talk about “the product” of a family $\{F_x\}_{x \in X}$. See also A.2.7 below and Remark A.2.2 above.

A.2.6. Example. If $\mathbf{B} = \mathbf{Set}$, the category of sets, and $\{F_1, F_2\}$ is a family of two sets then both Cartesian products $F_1 \times F_2$ and $F_2 \times F_1$ (together with the projection onto the factors F_1, F_2) are categorical products of $\{F_1, F_2\}$. The unique isomorphism $F_1 \times F_2 \rightarrow F_2 \times F_1$ swaps the elements of the ordered pairs.

If \mathbf{B} is the category \mathbf{Vect} of vector spaces over the reals and then the categorical product of $\{F_x\}_{x \in X}$ is the product vector space $\prod_{x \in X} F_x$. If X is finite, it is also the direct sum $\bigoplus_{x \in X} F_x$. If X is empty then the zero vector space $\{0\}$ has the right universal property and is considered the (categorical) product of $\{F_x\}_{x \in \emptyset}$:

$$\prod_{x \in \emptyset} F_x = \{0\}.$$

If X is infinite and $\mathbf{B} = \mathbf{FinVect}$ then the categorical product $\prod_{x \in X} F_x$ does not exist in $\mathbf{FinVect}$, since the product of infinitely many vector spaces it is not a finite dimensional vector space (the Cartesian product $\prod_{x \in \emptyset} F_x$ is a vector space, but it is not finite dimensional).

If \mathbf{B} is the category \mathbf{Euc} of Euclidean spaces and smooth maps and X is a finite set then the Cartesian product $\prod_{x \in X} F_x$ of Euclidean spaces is a Euclidean space. Hence finite categorical products exist in \mathbf{Euc} . If X is infinite then the categorical product $\prod_{x \in X} F_x$ does not exist in \mathbf{Euc} . If X is empty then the $\prod_{x \in \emptyset} F_x$ is the one point Euclidean space $\{0\}$.

If \mathbf{B} is the category \mathbf{Man} of finite dimensional manifolds and smooth maps and X is a finite set then the categorical product $\prod_{x \in X} F_x$ is the Cartesian product of manifolds. Just as for the categories of finite dimensional vector spaces and Euclidean spaces, if the set X is infinite then the categorical product $\prod_{x \in X} F_x$ does not exist in \mathbf{Man} .

A.2.7 (Categorical products are limits). To see that categorical products are limits think of a family of objects $\{F_x\}_{x \in X}$ in a category \mathbf{B} as a functor $F: \mathbf{X} \rightarrow \mathbf{B}$, $F(x) = F_x$, from the corresponding discrete category \mathbf{X} (q.v. Remark A.1.27). Then $\lim(F: \mathbf{X} \rightarrow \mathbf{B})$ has exactly the universal properties of the categorical product $\prod_{x \in X} F_x$.

A.2.8. Remark (The set of maps into a product is a product). Suppose we have a family of objects $\{F_x\}_{x \in X}$ in a category \mathbf{B} and its product $\prod_{x \in X} F_x$ exists in \mathbf{B} . Let $\{p_x: \prod_{x' \in X} F_{x'} \rightarrow F_x\}$ denote the family of the canonical projections. For any morphism $f: Y \rightarrow \prod_{x \in X} F_x$ in \mathbf{B} and any $x \in X$ we have a morphism $p_x \circ f: Y \rightarrow F_x$. This defines a map

$$\mathbf{p}_x: \text{Hom}_{\mathbf{B}}(Y, \prod_{x \in X} F_x) \rightarrow \text{Hom}_{\mathbf{B}}(Y, F_x), \quad \mathbf{p}_x(f) := p_x \circ f \equiv (p_x)_* f.$$

On the other hand, since arbitrary products exist in the category **Set** of sets, we have the product $\prod_{x \in X} \text{Hom}_B(Y, F_x)$. Denote its family of the canonical projections by $\{\pi_x: \prod_{x' \in X} \text{Hom}_B(Y, F_{x'}) \rightarrow \text{Hom}_B(Y, F_x)\}$. By the universal property of the product $\prod_{x \in X} \text{Hom}_B(Y, F_x)$ the maps $\{\mathfrak{p}_x\}_{x \in X}$ define a unique map

$$\mathfrak{p}: \text{Hom}_B(Y, \prod_{x \in X} F_x) \rightarrow \prod_{x \in X} \text{Hom}_B(Y, F_x).$$

Informally speaking the map \mathfrak{p} sends a map f to the tuple of maps $(p_x \circ f)_{x \in X} \in \prod_{x \in X} \text{Hom}_B(Y, F_x)$. By the universal property of the product $\{p_x: \prod_{x' \in X} F_{x'} \rightarrow F_x\}$ the map \mathfrak{p} is a bijection.

There is an even better way to think of about the universal property of the family $\{\mathfrak{p}_x: \text{Hom}_B(Y, \prod_{x' \in X} F_{x'}) \rightarrow \text{Hom}_B(Y, F_x)\}_{x \in X}$. Namely, $\text{Hom}_B(Y, \prod_{x \in X} F_x)$ together with the family $\{\mathfrak{p}_x\}_{x \in X}$ is a product of the family of sets $\{\text{Hom}_B(Y, F_x)\}_{x \in X}$:

$$\text{Hom}_B(Y, \prod_{x \in X} F_x) = \prod_{x \in X} \text{Hom}_B(Y, F_x).$$

Compare with A.2.4.

A.2.9 (Invariants are limits). Let $\rho: G \rightarrow GL(V)$ be a representation of a group G . Recall that the space of invariants, the space of G -fixed vectors, is

$$V^G := \{v \in V \mid \rho(g)v = v \text{ for all } g \in G\}.$$

We now interpret V^G together with its inclusion $V^G \hookrightarrow V$ as a limit of functor. View the group G as a category \mathbf{G} with one object (see Example A.1.13 above). Then the representation ρ defines a functor $\rho: \mathbf{G} \rightarrow \mathbf{Vect}$. The functor ρ sends the unique object $*$ of \mathbf{G} to V and an arrow $g \in \mathbf{G}_1 = G$ to $\rho(g) \in \text{Hom}_{\mathbf{Vect}}(V, V) = \text{Hom}_{\mathbf{Vect}}(\rho(*), \rho(*))$. By definition of a limit, the limit of $\rho: \mathbf{G} \rightarrow \mathbf{Vect}$ is a vector space U together with a map $p: U \rightarrow \rho(*) = V$ with the following universal property: any linear map $T: W \rightarrow V$ with

$$(A.2.10) \quad \rho(g) \circ T = T$$

for all $g \in G$ should factor through $p: U \rightarrow V$. Now (A.2.10) implies that for any $w \in W$

$$\rho(g)T(w) = T(w) \text{ for all } g \in G.$$

Thus T always factors through the inclusion $V^G \hookrightarrow V$ and therefore $\lim(\rho: \mathbf{G} \rightarrow \mathbf{Vect}) = V^G$.

A.2.11. Definition. Let \mathbf{A} be a groupoid whose collection of objects \mathbf{A}_0 is a set. A *representation* F of \mathbf{A} is a functor from \mathbf{A} to the category of vector spaces \mathbf{Vect} :

$$F: \mathbf{A} \rightarrow \mathbf{Vect}$$

A.2.12. Definition. The *space of invariants* of a representation $F: \mathbf{A} \rightarrow \mathbf{Vect}$ of a groupoid \mathbf{A} is the limit $\lim F$.

A.2.13. Remark. The space of invariants of $F: \mathbf{A} \rightarrow \mathbf{Vect}$ has the following concrete description: consider the product $\prod_{a \in \mathbf{A}_0} F(a)$ together with its canonical projections $p_b: \prod_{a \in \mathbf{A}_0} F(a) \rightarrow F(b)$. Then

$$U := \{v \in \prod_{a \in \mathbf{A}_0} F(a) \mid F(\sigma)p_a(v) = p_b(v) \text{ for all arrows } a \xrightarrow{\sigma} b \in \mathbf{A}_1\}$$

together with the projections $\varpi_a = p_a|_U: U \rightarrow F(a)$. To check this assertion just check the universal properties of the family $\{\varpi_a: U \rightarrow F(a)\}_{a \in \mathbf{A}_0}$.

If the groupoid \mathbf{A} is a disjoint union of groups $\bigsqcup_{a \in A} G_a$ then a representation $F: \mathbf{A} \rightarrow \mathbf{Vect}$ is a family of representations $\{\rho_a: G_a \rightarrow V_a\}_{a \in A}$, where $\rho_a = F|_{G_a}$ and $V_a = F(a)$. It is not hard to

check that the product $\prod_{a \in A} V_a^{G_a}$ together with the canonical maps $\{\prod_{a \in A} V_a^{G_a} \rightarrow V_a\}_{a \in A}$ has the universal property of the limit $\lim F$. Thus in this case

$$\lim(F: \bigsqcup_a G_a \rightarrow \mathbf{Vect}) = \prod_{a \in A} V_a^{G_a},$$

i.e., it is the product of the invariants of constituent representations.

A.2.14. Remark. Since any groupoid \mathbf{A} is equivalent to disjoint union of groups (Example A.1.35), it follows from Theorem 6.1.5 that the space of invariants of a representation of a groupoid is always the product of invariants of representations of groups. C.f. 2.10.17 and its proof.

We end the appendix by restating the definition of a limit of a functor in terms of cones and terminal objects. We will use the restatement in Section 6.

A.2.15. Definition (Cones). Let $F: \mathbf{A} \rightarrow \mathbf{B}$ be a functor. A *cone* over F is an object b of \mathbf{B} together with a collection of arrows $\{\xi_a: b \rightarrow F(a)\}_{a \in \mathbf{A}_0}$ so that

$$\begin{array}{ccc} & b & \\ \xi_a \swarrow & & \searrow \xi_{a'} \\ F(a) & \xrightarrow{F(f)} & F(a') \end{array}$$

commutes for any arrow $f \in \text{Hom}_{\mathbf{A}}(a, a')$.

A *morphism* $v: (b, \{\xi_a\}) \rightarrow (b', \{\xi'_a\})$ of cones over F is an arrow $v: b \rightarrow b'$ in \mathbf{B} making the triangle

$$\begin{array}{ccc} b & \xrightarrow{v} & b' \\ \xi_a \searrow & & \swarrow \xi'_{a'} \\ & F(a) & \end{array}$$

commute.

We have an evident category $\text{Cone}(F)$ of cones over a functor F .

A.2.16. Note that if $v: (b, \{\xi_a\}) \rightarrow (b', \{\xi'_a\})$ is a morphism of cones over F then the diagram

$$\begin{array}{ccc} b & \xrightarrow{v} & b \\ \xi_a \downarrow & \swarrow \xi_{a'} & \searrow \xi'_a \\ & F(a) & \\ & \xrightarrow{F(f)} & F(a') \\ & \swarrow \xi'_a & \searrow \xi_{a'} \\ & F(a') & \end{array}$$

automatically commutes.

A.2.17. Definition (Terminal object). An object τ of a category \mathbf{C} is *terminal* if for any object c of \mathbf{C} there is a unique arrow $x: c \rightarrow \tau$.

A.2.18. Example. Any one point set $\{*\}$ is terminal in the category \mathbf{Set} of sets: given a set X there is a unique function $f: X \rightarrow \{*\}$ that sends every element of X to $*$.

A.2.19. Definition (Limit, restated). A limit $\lim F$ of a functor $F: \mathbf{A} \rightarrow \mathbf{B}$ is a terminal object in the category $\text{Cone}(F)$ of cones over F .

A.2.20. Remark. Terminal objects need not exist in a given category. But any two terminal objects τ, τ' of a category \mathbf{C} are necessarily isomorphic. That is, terminal objects are unique up to a unique isomorphism. We leave a proof of this fact as an easy and standard exercise. Consequently limits are unique up to a unique isomorphism (compare A.2.2).

REFERENCES

- [1] E. M. Rogers and D. L. Kincaid. *Communication networks: toward a new paradigm for research*. New York Free Press, 1981.
- [2] B. W. Knight. Dynamics of encoding in a population of neurons. *Journal of General Physiology*, 59(6):734–766, 1972.
- [3] G. B. Ermentrout and N. Kopell. Frequency plateaus in a chain of weakly coupled oscillators, I. *SIAM Journal on Mathematical Analysis*, 15(2):215–237, 1984.
- [4] Y. Kuramoto. Collective synchronization of pulse-coupled oscillators and excitable units. *Physica D*, 50(1):15–30, May 1991.
- [5] Y. Kuramoto. *Chemical oscillations, waves, and turbulence*, volume 19 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, 1984.
- [6] David MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003.
- [7] Peter Dayan and L.F. Abbott. *Theoretical Neuroscience: Computational and Mathematical Modeling of Neural Systems*. MIT Press, 2001.
- [8] James M. Bower and Hamid Bolouri, editors. *Computational Modeling of Genetic and Biochemical Networks*. MIT Press, January 2001.
- [9] Ilya Shmulevich, Edward R. Dougherty, Seungchan Kim, and Wei Zhang. Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks. *Bioinformatics*, 18(2):261–274, 2002.
- [10] Elena R Alvarez-Buylla, Mariana Benítez, Enrique Balleza Dávila, Álvaro Chaos, Carlos Espinosa-Soto, and Pablo Padilla-Longoria. Gene regulatory network models for plant development. *Current Opinion in Plant Biology*, 10(1):83 – 91, 2007. Growth and Development / Edited by Cris Kuhlemeier and Neelima Sinha.
- [11] Darren James Wilkinson. *Stochastic modelling for systems biology*. Chapman & Hall/CRC Mathematical and Computational Biology Series. Chapman & Hall/CRC, Boca Raton, FL, 2006.
- [12] C. S. Peskin. *Mathematical aspects of heart physiology*. Courant Institute of Mathematical Sciences New York University, New York, 1975. Notes based on a course given at New York University during the year 1973/74, see <http://math.nyu.edu/faculty/peskin/heartnotes/index.html>.
- [13] John J. Tyson and James P. Keener. Singular perturbation theory of traveling waves in excitable media (a review). *Phys. D*, 32(3):327–361, 1988.
- [14] Raymond Kapral and Kenneth Showalter, editors. *Chemical Waves and Patterns*. Springer, 1994.
- [15] G. Bard Ermentrout and John Rinzel. Reflected waves in an inhomogeneous excitable medium. *SIAM Journal on Applied Mathematics*, 56(4):1107–1128, 1996.
- [16] James P. Keener and James Sneyd. *Mathematical Physiology*, volume 8 of *Interdisciplinary Applied Mathematics*. Elsevier, 1998.
- [17] Duncan J. Watts and Steven H. Strogatz. Collective dynamics of ‘small-world’ networks. *Nature*, 393(6684):440–442, Jun 4 1998.
- [18] Albert-László Barabási and Réka Albert. Emergence of scaling in random networks. *Science*, 286(5439):509–512, 1999.
- [19] Steven H. Strogatz. Exploring complex networks. *Nature*, 410:268–276, 8 March 2001.
- [20] N. Schwartz, R. Cohen, D. ben Avraham, A.-L. Barabási, and S. Havlin. Percolation in directed scale-free networks. *Phys. Rev. E* (3), 66(1):015104, 4, 2002.
- [21] Réka Albert and Albert-László Barabási. Statistical mechanics of complex networks. *Rev. Modern Phys.*, 74(1):47–97, 2002.
- [22] Albert-László Barabási. Emergence of scaling in complex networks. In *Handbook of graphs and networks*, pages 69–84. Wiley-VCH, Weinheim, 2003.
- [23] Albert-László Barabási, Zoltán N. Oltvai, and Stefan Wuchty. Characteristics of biological networks. In *Complex networks*, volume 650 of *Lecture Notes in Phys.*, pages 443–457. Springer, Berlin, 2004.
- [24] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: structure and dynamics. *Phys. Rep.*, 424(4-5):175–308, 2006.
- [25] Martin Golubitsky and Ian Stewart. Hopf bifurcation in the presence of symmetry. *Bull. Amer. Math. Soc. (N.S.)*, 11(2):339–342, 1984.
- [26] Martin Golubitsky and Ian Stewart. Hopf bifurcation in the presence of symmetry. *Arch. Rational Mech. Anal.*, 87(2):107–165, 1985.
- [27] Martin Golubitsky and Ian Stewart. Hopf bifurcation with dihedral group symmetry: coupled nonlinear oscillators. In *Multiparameter bifurcation theory (Arcata, Calif., 1985)*, volume 56 of *Contemp. Math.*, pages 131–173. Amer. Math. Soc., Providence, RI, 1986.
- [28] Martin Golubitsky and Ian Stewart. Symmetry and stability in Taylor-Couette flow. *SIAM J. Math. Anal.*, 17(2):249–288, 1986.

- [29] Martin Golubitsky and Ian Stewart. Generic bifurcation of Hamiltonian systems with symmetry. *Phys. D*, 24(1-3):391–405, 1987. With an appendix by Jerrold Marsden.
- [30] Martin Golubitsky, Ian Stewart, and David G. Schaeffer. *Singularities and groups in bifurcation theory. Vol. II*, volume 69 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1988.
- [31] M. Field, M. Golubitsky, and I. Stewart. Bifurcations on hemispheres. *J. Nonlinear Sci.*, 1(2):201–223, 1991.
- [32] Martin Golubitsky, Ian Stewart, and Benoit Dionne. Coupled cells: wreath products and direct products. In *Dynamics, bifurcation and symmetry (Cargèse, 1993)*, volume 437 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 127–138. Kluwer Acad. Publ., Dordrecht, 1994.
- [33] Michael Dellnitz, Martin Golubitsky, Andreas Hohmann, and Ian Stewart. Spirals in scalar reaction-diffusion equations. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 5(6):1487–1501, 1995.
- [34] Benoit Dionne, Martin Golubitsky, Mary Silber, and Ian Stewart. Time-periodic spatially periodic planforms in Euclidean equivariant partial differential equations. *Philos. Trans. Roy. Soc. London Ser. A*, 352(1698):125–168, 1995.
- [35] Benoit Dionne, Martin Golubitsky, and Ian Stewart. Coupled cells with internal symmetry. I. Wreath products. *Nonlinearity*, 9(2):559–574, 1996.
- [36] Benoit Dionne, Martin Golubitsky, and Ian Stewart. Coupled cells with internal symmetry. II. Direct products. *Nonlinearity*, 9(2):575–599, 1996.
- [37] Martin Golubitsky, Ian Stewart, Pietro-Luciano Buono, and J. J. Collins. A modular network for legged locomotion. *Phys. D*, 115(1-2):56–72, 1998.
- [38] Martin Golubitsky and Ian Stewart. Symmetry and pattern formation in coupled cell networks. In *Pattern formation in continuous and coupled systems (Minneapolis, MN, 1998)*, volume 115 of *IMA Vol. Math. Appl.*, pages 65–82. Springer, New York, 1999.
- [39] Martin Golubitsky and Ian Stewart. Symmetry and pattern formation in coupled cell networks. In *Pattern formation in continuous and coupled systems (Minneapolis, MN, 1998)*, volume 115 of *IMA Vol. Math. Appl.*, pages 65–82. Springer, New York, 1999.
- [40] Martin Golubitsky and Ian Stewart. *The symmetry perspective*, volume 200 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002. From equilibrium to chaos in phase space and physical space.
- [41] M. Golubitsky, E. Knobloch, and I. Stewart. Target patterns and spirals in planar reaction-diffusion systems. *J. Nonlinear Sci.*, 10(3):333–354, 2000.
- [42] Martin Golubitsky and Ian Stewart. Patterns of oscillation in coupled cell systems. In *Geometry, mechanics, and dynamics*, pages 243–286. Springer, New York, 2002.
- [43] Martin Golubitsky and Ian Stewart. *The symmetry perspective*, volume 200 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002. From equilibrium to chaos in phase space and physical space.
- [44] Ian Stewart, Martin Golubitsky, and Marcus Pivato. Symmetry groupoids and patterns of synchrony in coupled cell networks. *SIAM J. Appl. Dyn. Syst.*, 2(4):609–646 (electronic), 2003.
- [45] M. Golubitsky, M. Nicol, and I. Stewart. Some curious phenomena in coupled cell networks. *J. Nonlinear Sci.*, 14(2):207–236, 2004.
- [46] M. Golubitsky, M. Pivato, and I. Stewart. Interior symmetry and local bifurcation in coupled cell networks. *Dyn. Syst.*, 19(4):389–407, 2004.
- [47] Martin Golubitsky and Ian Stewart. Synchrony versus symmetry in coupled cells. In *EQUADIFF 2003*, pages 13–24. World Sci. Publ., Hackensack, NJ, 2005.
- [48] Martin Golubitsky, Ian Stewart, and Andrei Török. Patterns of synchrony in coupled cell networks with multiple arrows. *SIAM J. Appl. Dyn. Syst.*, 4(1):78–100 (electronic), 2005.
- [49] Martin Golubitsky and Ian Stewart. Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc. (N.S.)*, 43(3):305–364, 2006.
- [50] M. Golubitsky, K. Josić, and E. Shea-Brown. Winding numbers and average frequencies in phase oscillator networks. *J. Nonlinear Sci.*, 16(3):201–231, 2006.
- [51] Martin Golubitsky, Liejune Shiau, and Ian Stewart. Spatiotemporal symmetries in the disinaptic canal-neck projection. *SIAM J. Appl. Math.*, 67(5):1396–1417 (electronic), 2007.
- [52] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [53] R. E. Lee DeVille and Eugene Lerman. Dynamics on networks II. Combinatorial categories of modular discrete-time systems. in preparation.
- [54] R. E. Lee DeVille and Eugene Lerman. Dynamics on networks III. Combinatorial categories of modular stochastic dynamical systems. in preparation.
- [55] Paulo Tabuada and George J. Pappas. Quotients of fully nonlinear control systems. *SIAM J. Control Optim.*, 43(5):1844–1866 (electronic), 2005.

- [56] Michael J. Field. *Dynamics and symmetry*, volume 3 of *ICP Advanced Texts in Mathematics*. Imperial College Press, London, 2007.
- [57] Paolo Boldi and Sebastiano Vigna. Fibrations of graphs. *Discrete Math.*, 243(1-3):21–66, 2002.
- [58] Paolo Boldi, Violetta Lonati, Massimo Santini, and Sebastiano Vigna. Graph fibrations, graph isomorphism, and PageRank. *Theor. Inform. Appl.*, 40(2):227–253, 2006.
- [59] Philip J. Higgins. *Notes on categories and groupoids*. Van Nostrand Reinhold Co., London, 1971. Van Nostrand Reinhold Mathematical Studies, No. 32.
- [60] Sebastiano Vigna. <http://vigna.dsi.unimi.it/fibrations/>.
- [61] Allen Knutson and Terence Tao. Honeycombs and sums of Hermitian matrices. *Notices Amer. Math. Soc.*, 48(2):175–186, 2001.
- [62] Chi-Kwong Li and Yiu-Tung Poon. Principal submatrices of a Hermitian matrix. *Linear Multilinear Algebra*, 51(2):199–208, 2003.
- [63] Steve Awodey. *Category theory*, volume 49 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2006.

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